

Exact controllability for quasi-linear perturbations of KdV

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Abstract. We prove that the KdV equation on the circle remains exactly controllable in arbitrary time with localized control, for sufficiently small data, also in presence of quasi-linear perturbations, namely nonlinearities containing up to three space derivatives, having a Hamiltonian structure at the highest orders. We use a procedure of reduction to constant coefficients up to order zero (adapting [5]), classical Ingham inequality and HUM method to prove the controllability of the linearized operator. Then a fine version of the Nash-Moser implicit function theorem due to Hörmander can be applied as a “black box”. *MSC2010:* 35Q53, 35Q93.

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1 Introduction

A question in control theory for PDEs regards the persistence of controllability under perturbations. In this paper we study the effect of *quasi-linear* perturbations (namely nonlinearities containing derivatives of the highest order) on the controllability of the KdV equation. We consider equations of the form

$$u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = 0 \quad (1.1)$$

on the circle $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, with $t \in \mathbb{R}$, where $u = u(t, x)$ is real-valued, and \mathcal{N} is a given real-valued nonlinear function which is at least quadratic around $u = 0$. For solutions of small amplitude, (1.1) is a quasi-linear perturbation of the Airy equation $u_t + u_{xxx} = 0$, which is the linear part of KdV; then the KdV nonlinear term uu_x can be included in \mathcal{N} .

Motivated by a question, which was posed in [29], about the possibility of including the dependence on higher derivatives in nonlinear perturbations of KdV, equations of the form (1.1) have recently been studied in [5, 6, 7] in the context of KAM theory. In this paper we study (1.1) from the point of view of control theory, proving its exact controllability by means of an internal control, in arbitrary time, for sufficiently small data (Theorem 1.1).

Most of the known results about controllability of quasi-linear PDEs deal with first order quasi-linear hyperbolic systems of the form $u_t + A(u)u_x = 0$ (including quasi-linear wave, shallow water, and Euler equations), see for example Li and Zhang [35], Coron [17] (chapter 6.2, and see also the many references therein), Li and Rao [34], Coron, Glass and Wang [18], and recently Alabau-Boussouira, Coron and Olive [1]. Recent results for different kinds of quasi-linear PDEs are contained in Alazard, Baldi and Han-Kwan [3] on the internal controllability of 2D gravity-capillary water waves equations, and Alazard [2] on the boundary observability of 2D and 3D (fully nonlinear) gravity water waves. For a general introduction to the theory of control for PDEs see, for example, Lions [36], Micu

and Zuazua [37], Coron [17], while for important results in control for hyperbolic PDEs see, for example, Bardos, Lebeau and Rauch [8], Burq and Gérard [15], Burq and Zworski [16].

Regarding the KdV equation, the first controllability results are due to Zhang [43] and Russell [41]. Among recent results, we mention the work by Laurent, Rosier and Zhang [33] for large data. A beautiful review on the literature on control for KdV can be found in [40]. For more on KdV, see the rich survey [23] by Guan and Kuksin, and the many references therein.

1.1 Main result

We assume that the nonlinearity $\mathcal{N}(x, u, u_x, u_{xx}, u_{xxx})$ is at least quadratic around $u = 0$, namely the real-valued function $\mathcal{N} : \mathbb{T} \times \mathbb{R}^4 \rightarrow \mathbb{R}$ satisfies

$$|\mathcal{N}(x, z_0, z_1, z_2, z_3)| \leq C|z|^2 \quad \forall z = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4, \quad |z| \leq 1. \quad (1.2)$$

We assume that the dependence of \mathcal{N} on u_{xx}, u_{xxx} is Hamiltonian, while no structure is required on its dependence on u, u_x . More precisely, we assume that

$$\mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = \mathcal{N}_1(x, u, u_x, u_{xx}, u_{xxx}) + \mathcal{N}_0(x, u, u_x) \quad (1.3)$$

where

$$\begin{aligned} \mathcal{N}_1(x, u, u_x, u_{xx}, u_{xxx}) &= \partial_x \{(\partial_u \mathcal{F})(x, u, u_x)\} - \partial_{xx} \{(\partial_{u_x} \mathcal{F})(x, u, u_x)\} \\ &\text{for some function } \mathcal{F} : \mathbb{T} \times \mathbb{R}^2 \rightarrow \mathbb{R}. \end{aligned} \quad (1.4)$$

Note that the case $\mathcal{N} = \mathcal{N}_1$, $\mathcal{N}_0 = 0$ corresponds to the Hamiltonian equation $\partial_t u = \partial_x \nabla H(u)$ where the Hamiltonian is

$$H(u) = \frac{1}{2} \int_{\mathbb{T}} u_x^2 dx + \int_{\mathbb{T}} \mathcal{F}(x, u, u_x) dx \quad (1.5)$$

and ∇ denotes the $L^2(\mathbb{T})$ -gradient. The unperturbed KdV is the case $\mathcal{F} = -\frac{1}{6}u^3$.

Notation. For periodic functions $u(x)$, $x \in \mathbb{T}$, we expand $u(x) = \sum_{n \in \mathbb{Z}} u_n e^{inx}$, and, for $s \in \mathbb{R}$, we consider the standard Sobolev space of periodic functions

$$H_x^s := H^s(\mathbb{T}, \mathbb{R}) := \{u : \mathbb{T} \rightarrow \mathbb{R} : \|u\|_s < \infty\}, \quad \|u\|_s^2 := \sum_{n \in \mathbb{Z}} |u_n|^2 \langle n \rangle^{2s}, \quad (1.6)$$

where $\langle n \rangle := (1 + n^2)^{\frac{1}{2}}$. We consider the space $C([0, T], H_x^s)$ of functions $u(t, x)$ that are continuous in time with values in H_x^s . We will use the following notation for the standard norm in $C([0, T], H_x^s)$:

$$\|u\|_{T,s} := \|u\|_{C([0,T], H_x^s)} := \sup_{t \in [0,T]} \|u(t)\|_s. \quad (1.7)$$

Theorem 1.1 (Exact controllability). *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be a nonempty open set. There exist positive universal constants r, σ, s_1 such that, if \mathcal{N} in (1.1) is of class C^r in its arguments and satisfies (1.2), (1.3), (1.4), then there exists a positive constant δ_* depending on T, ω, \mathcal{N} with the following property.*

Let $u_{in}, u_{end} \in H^{s_1}(\mathbb{T}, \mathbb{R})$ with

$$\|u_{in}\|_{s_1} + \|u_{end}\|_{s_1} \leq \delta_*.$$

Then there exists a function $f(t, x)$ satisfying

$$f(t, x) = 0 \quad \text{for all } x \notin \omega, \text{ for all } t \in [0, T],$$

belonging to $C([0, T], H_x^s) \cap C^1([0, T], H_x^{s-3}) \cap C^2([0, T], H_x^{s-6})$ for all $s < s_1$, such that the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = f & \forall (t, x) \in [0, T] \times \mathbb{T} \\ u(0, x) = u_{in}(x) \end{cases} \quad (1.8)$$

has a unique solution $u(t, x)$ belonging to $C([0, T], H_x^s) \cap C^1([0, T], H_x^{s-3}) \cap C^2([0, T], H_x^{s-6})$ for all $s < s_1$, which satisfies

$$u(T, x) = u_{end}(x). \quad (1.9)$$

Moreover, for all $s < s_1$,

$$\begin{aligned} \|u, f\|_{C([0, T], H_x^s)} + \|\partial_t u, \partial_t f\|_{C([0, T], H_x^{s-3})} + \|\partial_{tt} u, \partial_{tt} f\|_{C([0, T], H_x^{s-6})} \\ \leq C_s (\|u_{in}\|_{s_1} + \|u_{end}\|_{s_1}) \end{aligned} \quad (1.10)$$

for some $C_s > 0$ depending on $s, T, \omega, \mathcal{N}$.

Remark 1.2. In Theorem 1.1 there is an arbitrarily small loss of regularity: if the initial and final data u_{in}, u_{end} have Sobolev regularity $H_x^{s_1}$, then the control f and the solution u are continuous in time with values in H_x^s for all $s < s_1$. Such loss of regularity is in some sense fictitious: it is due to our choice of working with standard Sobolev spaces, but it could be avoided by working with the (slightly “worse-looking”) weak spaces E'_a introduced by Hörmander in [26] (see Section B). What we actually prove is that, if the initial and final data are in the weak space $(H_x^{s_1})'$ (i.e. the weak version *à la* Hörmander [26] of the Sobolev space $H_x^{s_1}$), then f and u are continuous in time with values in the same space $(H_x^{s_1})'$. \square

Remark 1.3. Our proof of Theorem 1.1 does not use results of existence and uniqueness for the Cauchy problem (1.8). On the contrary, our method directly proves local existence and uniqueness for (1.8) (see Theorem 1.4). This situation occurs quite often in control problems (see Remark 4.12 in [17]). \square

1.2 Description of the proof

It would be natural to try to solve the control problem (1.8)-(1.9) using a fixed point argument or the usual implicit function theorem. However, this seems to be impossible because of the presence of three derivatives in the nonlinear term. A similar difficulty was overcome in [3] by using a suitable nonlinear iteration scheme adapted to quasi-linear problems. Such a nonlinear scheme requires to solve a linear control problem with variable

coefficients at each step of the iteration, with no loss of regularity with respect to the coefficients (i.e., the solution must have the same regularity as the coefficients). In [3] this is achieved by means of para-differential calculus, together with linear transformations, Ingham-type inequalities and the Hilbert uniqueness method.

As an alternative method, in this paper we use a Nash-Moser implicit function theorem. The Nash-Moser approach also demands to solve a linear control problem with variable coefficients, but it has the advantage of requiring weaker estimates, allowing a loss of regularity with respect to the coefficients. The proof of such weaker estimates is easier to obtain, and it does not require the use of powerful techniques like para-differential calculus. In this sense our Nash-Moser method is alternative to the method in [3] (for a discussion about pseudo- and para-differential calculus in connection with the Nash-Moser theorem, see, for example, Hörmander [27], Alinhac and Gérard [4]). Another advantage of our approach is that the Nash-Moser theorem can be used “*as a black box*”, without entering the details of the nonlinear scheme. On the other hand, the result that we obtain with the Nash-Moser method is slightly weaker than the one in [3] regarding the regularity of the solution of the nonlinear control problem with respect to the regularity of the data: the arbitrarily small loss of regularity in Theorem 1.1 is discussed in Remark 1.2, while Theorem 1.1 of [3] has no loss of regularity also in the standard Sobolev spaces.

Nash-Moser schemes in control problems for PDEs have been used by Beauchard, Coron, Alabau-Boussouira, Olive in [9, 11, 10, 1]. A discussion about Nash-Moser as a method to overcome the problem of the loss of derivatives in the context of controllability for PDEs can be found in [17, section 4.2.2]. In [12] Beauchard and Laurent were able to avoid the use of the Nash-Moser theorem in semilinear control problems thanks to some regularizing effect. We remark that Theorem 1.1 could also be proved without Nash-Moser (for example, by adapting the method of [3]). However, for our quasi-linear problem (1.1), we find the Nash-Moser approach convenient.

Now we describe our method in more detail. Given a nonempty open set $\omega \subset \mathbb{T}$, we first fix a C^∞ function $\chi_\omega(x)$ with values in the interval $[0, 1]$ which vanishes outside ω , and takes value $\chi_\omega = 1$ on a nonempty open subset of ω . Thus, given initial and final data u_{in}, u_{end} , we look for u, f that solve

$$\begin{cases} P(u) = \chi_\omega f \\ u(0) = u_{in} \\ u(T) = u_{end} \end{cases} \quad (1.11)$$

where

$$P(u) := u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}). \quad (1.12)$$

We define

$$\Phi(u, f) := \begin{pmatrix} P(u) - \chi_\omega f \\ u(0) \\ u(T) \end{pmatrix} \quad (1.13)$$

so that problem (1.11) is written as

$$\Phi(u, f) = (0, u_{in}, u_{end}).$$

The crucial assumption to verify in order to apply any Nash-Moser theorem is the existence of a right inverse of the linearized operator. The linearized operator $\Phi'(u, f)[h, \varphi]$ at the

point (u, f) in the direction (h, φ) is

$$\Phi'(u, f)[h, \varphi] := \begin{pmatrix} P'(u)[h] - \chi_\omega \varphi \\ h(0) \\ h(T) \end{pmatrix}. \quad (1.14)$$

Thus we have to prove that, given any (u, f) and any $g := (g_1, g_2, g_3)$ in suitable function spaces, there exists (h, φ) such that

$$\Phi'(u, f)[h, \varphi] = g. \quad (1.15)$$

Moreover we have to estimate (h, φ) in terms of u, f, g in a “tame” way (an estimate is said to be tame when it is linear in the highest norms: see (B.10) and (4.41)).

Problem (1.15) is a linear control problem. We observe that the linearized operator $P'(u)[h]$ is a differential operator having variable coefficients also at the highest order (which is a consequence of linearizing a *quasi-linear* PDE). Explicitly, it has the form

$$P'(u)[h] = \partial_t h + (1 + a_3(t, x))\partial_{xxx} h + a_2(t, x)\partial_{xx} h + a_1(t, x)\partial_x h + a_0(t, x)h.$$

We solve (1.15) in Theorem 4.5. Note that the choice of the function spaces is not given a priori: to fix a suitable functional setting is part of the problem.

Theorem 4.5 is proved by adapting a procedure of reduction to constant coefficients developed in [5, 6]. Such a procedure conjugates $P'(u)$ to an operator \mathcal{L}_5 (see (2.57)) having constant coefficients up to a bounded remainder. This conjugation is achieved by means of changes of the space variable, reparametrization of time, multiplication operators, and Fourier multipliers. Using Ingham inequality and a perturbation argument we prove the observability of \mathcal{L}_5 . Then we prove the observability of $P'(u)$ exploiting the explicit formulas of the transformations that conjugate $P'(u)$ to \mathcal{L}_5 . The linear control problem (1.15) is solved in L_x^2 by the HUM (Hilbert uniqueness method). Then further regularity of the solution (h, φ) of (1.15) is proved by adapting an argument used by Dehman-Lebeau [19], Laurent [32], and [3]. To conclude the proof of Theorem 1.1 we apply a Nash-Moser theorem in the version of Hörmander [26].

This method is not confined to KdV, and it could be applied to prove controllability of other quasi-linear evolution PDEs.

The use of Ingham-type inequalities and HUM is classical in control theory (see, for example, [25, 37, 31, 28] for Ingham and [36, 37, 17, 30] for HUM). As mentioned above, the Nash-Moser theorem has also been used in control theory (see, for example, [9, 11, 10, 1]). It was first introduced by Nash [39], then several refinements were developed afterwards, see for example Moser [38], Zehnder [42], Hamilton [24], Gromov [22], Hörmander [26, 27], and, recently, Berti, Bolle, Corsi and Procesi [13, 14], Ekeland and Séré [20, 21]. In our proof we use Hörmander’s version [26] of the Nash-Moser theorem because, for our problem, it seems to be the best one concerning the loss of regularity of the solution with respect to the regularity of the data (see also Remark 1.2).

1.3 Byproduct: a local existence and uniqueness result

As a byproduct, with the same technique and no extra work, we have the following existence and uniqueness theorem for the Cauchy problem of the quasi-linear PDE (1.1).

Theorem 1.4 (Local existence and uniqueness). *There exist positive universal constants r, σ, s_0 such that, if \mathcal{N} in (1.1) is of class C^r in its arguments and satisfies (1.2), (1.3), (1.4), then the following property holds. For all $T > 0$ there exists $\delta_* > 0$ such that for all $u_{in} \in H_x^{s_0}$, $f \in C([0, T], H_x^{s_0}) \cap C^1([0, T], H_x^{s_0-6})$ (possibly $f = 0$) satisfying*

$$\|u_{in}\|_{s_0} + \|f\|_{T, s_0} + \|\partial_t f\|_{T, s_0-6} \leq \delta_* , \quad (1.16)$$

the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = f, & (t, x) \in [0, T] \times \mathbb{T} \\ u(0, x) = u_{in}(x) \end{cases} \quad (1.17)$$

has one and only one solution $u \in C([0, T], H_x^s) \cap C^1([0, T], H_x^{s-3}) \cap C^2([0, T], H_x^{s-6})$ for all $s < s_0$. Moreover, for all $s < s_0$,

$$\begin{aligned} & \|u\|_{C([0, T], H_x^s)} + \|\partial_t u\|_{C([0, T], H_x^{s-3})} + \|\partial_{tt} u\|_{C([0, T], H_x^{s-6})} \\ & \leq C_s \left(\|u_{in}\|_{s_0} + \|f\|_{C([0, T], H_x^{s_0})} + \|\partial_t f\|_{C([0, T], H_x^{s_0-6})} \right) \end{aligned} \quad (1.18)$$

for some $C_s > 0$ depending on s, T, \mathcal{N} .

Remark 1.5. Theorem 1.4 is not sharp: we expect that better results for the Cauchy problem (1.17) can be proved by using a para-differential approach. \square

Remark 1.6. The loss of regularity in Theorem 1.4 is of the same type as the one in Theorem 1.1, see the discussion in Remark 1.2. \square

1.4 Organization of the paper

In Section 2 we describe the transformations that conjugate the linearized operator $P'(u)$ to constant coefficients up to a bounded remainder, and we give quantitative estimates on these transformations. In Section 3 we exploit these results to prove the observability of $P'(u)$. In Section 4 we use observability to solve the linear control problem (1.15) via HUM (Theorem 4.5) and we fix suitable function spaces (4.36)-(4.37). In Section 5 we prove Theorems 1.1 and 1.4 by applying Hörmander's Nash-Moser theorem. In Appendix A we prove well-posedness with tame estimates for all the linear operators involved in the reduction procedure. These well-posedness results are used many times along the Sections 3, 4, 5. Finally, in Appendix B we recall the statement of the Nash-Moser theorem by Hörmander, as given in [26].

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2 Reduction of the linearized operator to constant coefficients

In this section we consider some changes of variables that conjugate the linearized operator to constant coefficients up to a bounded remainder. This reduction procedure closely follows the analysis in [5] and [6], with some adaptations.

The linearized operator $P'(u)$ is

$$P'(u)[h] = \partial_t h + (1 + a_3)\partial_{xxx}h + a_2\partial_{xx}h + a_1\partial_xh + a_0h, \quad (2.1)$$

where the coefficients $a_i = a_i(t, x)$, $i = 0, \dots, 3$ are real-valued functions of $(t, x) \in [0, T] \times \mathbb{T}$, depending on u by

$$a_i = a_i(u) := (\partial_{z_i}\mathcal{N})(x, u, u_x, u_{xx}, u_{xxx}), \quad i = 0, \dots, 3 \quad (2.2)$$

(recall the notation $\mathcal{N} = \mathcal{N}(x, z_0, z_1, z_2, z_3)$). Note that $a_2 = 2\partial_x a_3$ because of the Hamiltonian structure of the component \mathcal{N}_1 of the nonlinearity (see (1.3)-(1.4)). The coefficients a_i satisfy the following tame estimates (as a consequence of the classical tame estimates for composition of functions).

Lemma 2.1. *Let $\mathcal{N} \in C^r(\mathbb{T} \times \mathbb{R}^4, \mathbb{R})$ satisfying (1.2). For all $1 \leq s \leq r - 3$, and for all $u \in C^2([0, T], H_x^{s+3})$ such that $\|u, \partial_t u, \partial_{tt} u\|_{T,4} \leq 1$, the coefficients $a_i(u)$ satisfy*

$$\|a_i(u), \partial_t a_i(u), \partial_{tt} a_i(u)\|_{T,s} \leq C_s \|u, \partial_t u, \partial_{tt} u\|_{T,s+3}, \quad i = 0, 1, 2, 3, \quad (2.3)$$

where C_s depends on s .

Now we apply the reduction procedure to any linear operator of the form (2.1) where

$$a_2(t, x) = c\partial_x a_3(t, x) \quad (2.4)$$

for some constant $c \in \mathbb{R}$ (note that $P'(u)$ has $c = 2$ because of the Hamiltonian structure of \mathcal{N}_1). Regarding the loss of regularity with respect to the space variable x , the estimates in the sequel will be not sharp. In the whole section we consider $T > 0$ fixed, and, unless otherwise specified, all the constants may depend on T .

Remark 2.2. Given a linear operator \mathcal{L}_0 of the form (2.1), define the operator \mathcal{L}_0^* as

$$\mathcal{L}_0^*h := -\partial_t h - \partial_{xxx}\{(1 + a_3)h\} + \partial_{xx}(a_2h) - \partial_x(a_1h) + a_0h. \quad (2.5)$$

Note that $-\mathcal{L}_0^*$ is still an operator of the form (2.1), namely

$$-\mathcal{L}_0^* = \partial_t + (1 + a_3^*)\partial_{xxx} + a_2^*\partial_{xx} + a_1^*\partial_x + a_0^* \quad (2.6)$$

with

$$\begin{aligned} a_3^* &:= a_3, & a_2^* &:= 3(a_3)_x - a_2, \\ a_1^* &:= 3(a_3)_{xx} - 2(a_2)_x + a_1, & a_0^* &:= (a_3)_{xxx} - (a_2)_{xx} + (a_1)_x - a_0. \end{aligned} \quad (2.7)$$

It follows from (2.6), (2.7) that if \mathcal{L}_0 satisfies (2.4), then also $-\mathcal{L}_0^*$ satisfies (2.4) (with a different constant), namely $a_2^* = (3 - c)\partial_x a_3^*$. In particular, if \mathcal{L}_0 satisfies (2.4) with $c = 2$ (which is the case if $\mathcal{L}_0 = P'(u)$), then $-\mathcal{L}_0^*$ satisfies (2.4) with $c = 1$. \square

2.1 Step 1. Change of the space variable

We consider a t -dependent family of diffeomorphisms of the circle \mathbb{T} of the form

$$y = x + \beta(t, x), \quad (2.8)$$

where β is a real-valued function, 2π periodic in x , defined for $t \in [0, T]$, with $|\beta_x(t, x)| \leq 1/2$ for all $(t, x) \in [0, T] \times \mathbb{T}$. We define the linear operator

$$(\mathcal{A}h)(t, x) := h(t, x + \beta(t, x)). \quad (2.9)$$

The operator \mathcal{A} is invertible, with inverse \mathcal{A}^{-1} , transpose \mathcal{A}^T (transpose with respect to the usual L_x^2 -scalar product) and inverse transpose \mathcal{A}^{-T} given by

$$\begin{aligned} (\mathcal{A}^{-1}v)(t, y) &= v(t, y + \tilde{\beta}(t, y)), & (\mathcal{A}^T v)(t, y) &= (1 + \tilde{\beta}_y(t, y)) v(t, y + \tilde{\beta}(t, y)), \\ (\mathcal{A}^{-T}h)(t, x) &= (1 + \beta_x(t, x)) h(t, x + \beta(t, x)) \end{aligned} \quad (2.10)$$

where $y \mapsto y + \tilde{\beta}(t, y)$ is the inverse diffeomorphism of (2.8), namely

$$x = y + \tilde{\beta}(t, y) \iff y = x + \beta(t, x). \quad (2.11)$$

Given the operator

$$\mathcal{L}_0 := \partial_t + (1 + a_3(t, x))\partial_{xxx} + a_2(t, x)\partial_{xx} + a_1(t, x)\partial_x + a_0(t, x), \quad (2.12)$$

with $a_2(t, x) = c\partial_x a_3(t, x)$ we calculate the conjugate $\mathcal{A}^{-1}\mathcal{L}_0\mathcal{A}$. The conjugate $\mathcal{A}^{-1}a\mathcal{A}$ of any multiplication operator $a : h(t, x) \mapsto a(t, x)h(t, x)$ is the multiplication operator $(\mathcal{A}^{-1}a)$ that maps $v(t, y) \mapsto (\mathcal{A}^{-1}a)(t, y)v(t, y)$. By conjugation, the differential operators become

$$\mathcal{A}^{-1}\partial_t\mathcal{A} = \partial_t + (\mathcal{A}^{-1}\beta_t)\partial_y, \quad \mathcal{A}^{-1}\partial_x\mathcal{A} = \{\mathcal{A}^{-1}(1 + \beta_x)\}\partial_y$$

then $\mathcal{A}^{-1}\partial_{xx}\mathcal{A} = (\mathcal{A}^{-1}\partial_x\mathcal{A})(\mathcal{A}^{-1}\partial_x\mathcal{A})$, and similarly for the conjugate of ∂_{xxx} . We calculate

$$\mathcal{L}_1 := \mathcal{A}^{-1}\mathcal{L}_0\mathcal{A} = \partial_t + a_4(t, y)\partial_{yyy} + a_5(t, y)\partial_{yy} + a_6(t, y)\partial_y + a_7(t, y) \quad (2.13)$$

where

$$\begin{aligned} a_4 &= \mathcal{A}^{-1}\{(1 + a_3)(1 + \beta_x)^3\}, & a_5 &= \mathcal{A}^{-1}\{a_2(1 + \beta_x)^2 + 3(1 + a_3)\beta_{xx}(1 + \beta_x)\}, \\ a_6 &= \mathcal{A}^{-1}\{\beta_t + (1 + a_3)\beta_{xxx} + a_2\beta_{xx} + a_1(1 + \beta_x)\}, & a_7 &= \mathcal{A}^{-1}a_0. \end{aligned} \quad (2.14)$$

We look for $\beta(t, x)$ such that the coefficient $a_4(t, y)$ of the highest order derivative ∂_{yyy} in (2.13) does not depend on y , namely $a_4(t, y) = b(t)$ for some function $b(t)$ of t only. This is equivalent to

$$(1 + a_3(t, x))(1 + \beta_x(t, x))^3 = b(t), \quad (2.15)$$

namely

$$\beta_x = \rho_0, \quad \rho_0(t, x) := b(t)^{1/3}(1 + a_3(t, x))^{-1/3} - 1. \quad (2.16)$$

The equation (2.16) has a solution β , periodic in x , if and only if $\int_{\mathbb{T}} \rho_0(t, x) dx = 0$ for all t . This condition uniquely determines

$$b(t) = \left(\frac{1}{2\pi} \int_{\mathbb{T}} (1 + a_3(t, x))^{-\frac{1}{3}} dx \right)^{-3}. \quad (2.17)$$

Then we fix the solution (with zero average) of (2.16),

$$\beta(t, x) := (\partial_x^{-1} \rho_0)(t, x), \quad (2.18)$$

where $\partial_x^{-1} h$ is the primitive of h with zero average in x (defined in Fourier). We have conjugated \mathcal{L}_0 to

$$\mathcal{L}_1 = \mathcal{A}^{-1} \mathcal{L}_0 \mathcal{A} = \partial_t + a_4(t) \partial_{yyy} + a_5(t, y) \partial_{yy} + a_6(t, y) \partial_y + a_7(t, y), \quad (2.19)$$

where $a_4(t) := b(t)$ is defined in (2.17).

We prove here some bounds that will be used later.

Lemma 2.3. *There exist positive constants σ, δ_* with the following properties. Let $s \geq 0$, and let $a_3(t, x), a_2(t, x), a_1(t, x), a_0(t, x)$ be four functions with $a_2 = c \partial_x a_3$ for some $c \in \mathbb{R}$. Moreover, assume $\partial_{tt} a_3, \partial_t a_3, a_3, \partial_t a_1, a_1, a_0 \in C([0, T], H_x^{s+\sigma})$. Let*

$$\delta(\mu) := \|\partial_{tt} a_3, \partial_t a_3, a_3, \partial_t a_1, a_1, a_0\|_{T, \mu+\sigma} \quad \forall \mu \in [0, s]. \quad (2.20)$$

If $\delta(0) \leq \delta_$, then the operator \mathcal{A} defined in (2.9) belongs to $C([0, T], \mathcal{L}(H_x^\mu))$ for all $\mu \in [0, s]$ and satisfies*

$$\|\mathcal{A}h\|_{T, \mu} \leq C_\mu (\|h\|_{T, \mu} + \delta(\mu) \|h\|_{T, 0}) \quad \forall h \in C([0, T], H_x^\mu), \quad (2.21)$$

for some positive C_μ depending on μ . The inverse operator \mathcal{A}^{-1} , the transpose \mathcal{A}^T and the inverse transpose \mathcal{A}^{-T} all satisfy the same estimate (2.21) as \mathcal{A} .

The functions $a_4(t) = b(t)$, $a_5(t, y)$, $a_6(t, y)$, $a_7(t, y)$, $\beta(t, x)$, $\tilde{\beta}(t, y)$ defined in (2.14), (2.17), (2.18), (2.11) belong to $C([0, T], H_x^\mu)$ for all $\mu \in [0, s]$ and satisfy

$$\|\beta, \tilde{\beta}, a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T, \mu} + |a_4 - 1, a_4'|_T \leq C_\mu \delta(\mu). \quad (2.22)$$

Finally, the coefficient $a_5(t, y)$ satisfies

$$\int_{\mathbb{T}} a_5(t, y) dy = 0 \quad \forall t \in [0, T]. \quad (2.23)$$

Proof. The proof of (2.21) and (2.22) is a straightforward application of the standard tame estimates for the product and composition of functions.

In order to prove (2.23), we use the definition of $b(t)$ in (2.17), the equality $a_2 = c \partial_x a_3$, and the change of variables (2.11), and we compute

$$\begin{aligned} \int_{\mathbb{T}} a_5(t, y) dy &= \int_{\mathbb{T}} [a_2(1 + \beta_x)^2 + 3(1 + a_3)\beta_{xx}(1 + \beta_x)](1 + \beta_x) dx = \\ &= b(t) \left\{ c \int_{\mathbb{T}} \frac{\partial_x a_3(t, x)}{1 + a_3(t, x)} dx + 3 \int_{\mathbb{T}} \frac{\beta_{xx}(t, x)}{1 + \beta_x(t, x)} dx \right\} = \\ &= b(t) \left\{ c \int_{\mathbb{T}} \partial_x \ln(1 + a_3(t, x)) dx + 3 \int_{\mathbb{T}} \partial_x \ln(1 + \beta_x(t, x)) dx \right\} = 0. \quad \square \end{aligned}$$

2.2 Step 2. Time reparametrization

The goal of this section is to obtain a constant coefficient instead of $a_4(t)$. We consider a diffeomorphism $\psi : [0, T] \rightarrow [0, T]$ which gives the change of the time variable

$$\psi(t) = \tau \quad \Leftrightarrow \quad t = \psi^{-1}(\tau), \quad (2.24)$$

with $\psi(0) = 0$ and $\psi(T) = T$. We define

$$(\mathcal{B}h)(t, y) := h(\psi(t), y), \quad (\mathcal{B}^{-1}v)(\tau, y) := v(\psi^{-1}(\tau), y). \quad (2.25)$$

By conjugation, the differential operators become

$$\mathcal{B}^{-1}\partial_t\mathcal{B} = \rho(\tau)\partial_\tau, \quad \mathcal{B}^{-1}\partial_y\mathcal{B} = \partial_y, \quad \rho := \mathcal{B}^{-1}(\psi'), \quad (2.26)$$

and therefore (2.19) is conjugated to

$$\mathcal{B}^{-1}\mathcal{L}_1\mathcal{B} = \rho\partial_\tau + (\mathcal{B}^{-1}a_4)\partial_{yyy} + (\mathcal{B}^{-1}a_5)\partial_{yy} + (\mathcal{B}^{-1}a_6)\partial_y + (\mathcal{B}^{-1}a_7). \quad (2.27)$$

We look for ψ such that the (variable) coefficients of the highest order derivatives (∂_τ and ∂_{yyy}) are proportional, namely

$$(\mathcal{B}^{-1}a_4)(\tau) = m\rho(\tau) = m(\mathcal{B}^{-1}(\psi'))(\tau) \quad (2.28)$$

for some constant $m \in \mathbb{R}$. Since \mathcal{B} is invertible, this is equivalent to requiring that

$$a_4(t) = m\psi'(t). \quad (2.29)$$

Integrating on $[0, T]$ determines the value of the constant m , and then we fix ψ :

$$m := \frac{1}{T} \int_0^T a_4(t) dt, \quad \psi(t) := \frac{1}{m} \int_0^t a_4(s) ds. \quad (2.30)$$

With this choice of ψ we get

$$\mathcal{B}^{-1}\mathcal{L}_1\mathcal{B} = \rho\mathcal{L}_2, \quad \mathcal{L}_2 := \partial_\tau + m\partial_{yyy} + a_8(\tau, y)\partial_{yy} + a_9(\tau, y)\partial_y + a_{10}(\tau, y), \quad (2.31)$$

where

$$\begin{aligned} a_8(\tau, y) &:= \frac{1}{\rho(\tau)} (\mathcal{B}^{-1}a_5)(\tau, y), & a_9(\tau, y) &:= \frac{1}{\rho(\tau)} (\mathcal{B}^{-1}a_6)(\tau, y), \\ a_{10}(\tau, y) &:= \frac{1}{\rho(\tau)} (\mathcal{B}^{-1}a_7)(\tau, y). \end{aligned} \quad (2.32)$$

Note that for all $\tau \in [0, T]$ one has

$$\int_{\mathbb{T}} a_8(\tau, y) dy = \frac{1}{(\mathcal{B}^{-1}\psi')(\tau)} \int_{\mathbb{T}} (\mathcal{B}^{-1}a_5)(\tau, y) dy = \frac{1}{\psi'(t)} \int_{\mathbb{T}} a_5(t, y) dy = 0. \quad (2.33)$$

By straightforward calculations, we prove the following lemma.

Lemma 2.4. *There exists $\delta_* > 0$ with the following properties. Let $a_4 \in C([0, T], \mathbb{R})$ with $|a_4(t) - 1| \leq \delta_*$ for all $t \in [0, T]$. Then the operator \mathcal{B} defined in (2.25), (2.30) is an invertible isometry of $C([0, T], H_x^s)$ for all $s \geq 0$, namely*

$$\|\mathcal{B}h\|_{T,s} = \|h\|_{T,s} \quad \forall h \in C([0, T], H_x^s), \quad s \geq 0. \quad (2.34)$$

Moreover there exists a positive constant σ with the following property. Let $a_4 \in C^1([0, T], \mathbb{R})$, with $|a_4(t) - 1| \leq \delta_*$ and $|a_4'(t)| \leq 1$ for all $t \in [0, T]$. Let $s \geq 0$, and $a_5, \partial_t a_5, a_6, \partial_t a_6, a_7 \in C([0, T], H_x^s)$ with $\int_{\mathbb{T}} a_5(t, y) dy = 0$ for all $t \in [0, T]$. Then the functions $a_8(t, x)$, $a_9(t, x)$, $a_{10}(t, x)$, $\psi(t)$, $\rho(t)$ and the constant m defined in (2.32), (2.30), (2.26) satisfy

$$|m - 1| + |\psi' - 1, \rho - 1|_T + \|a_8, \partial_\tau a_8, a_9, \partial_\tau a_9, a_{10}\|_{T,s} \leq C \|a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T,s} \quad (2.35)$$

where C is independent of s . Moreover one has

$$\int_{\mathbb{T}} a_8(\tau, y) dy = 0 \quad \forall \tau \in [0, T]. \quad (2.36)$$

2.3 Step 3. Multiplication

In this section we eliminate the term $a_8(\tau, y)\partial_{yy}$ from the operator \mathcal{L}_2 defined in (2.31). To this end, we consider the multiplication operator \mathcal{M} defined as

$$\mathcal{M}h(\tau, y) := q(\tau, y)h(\tau, y) \quad (2.37)$$

with $q : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$. We compute

$$\mathcal{M}^{-1}\mathcal{L}_2\mathcal{M} = \partial_\tau + m\partial_{yyy} + a_{11}(\tau, y)\partial_{yy} + a_{12}(\tau, y)\partial_y + a_{13}(\tau, y) \quad (2.38)$$

with

$$a_{11} := a_8 + \frac{3mq_y}{q}, \quad a_{12} := a_9 + \frac{2a_8q_y + 3mq_{yy}}{q}, \quad a_{13} := \frac{\mathcal{L}_2 q}{q}. \quad (2.39)$$

We want to choose q such that $a_{11} = 0$, which is equivalent to

$$3mq_y + a_8q = 0. \quad (2.40)$$

Thanks to (2.36), equation (2.40) admits the space-periodic solution

$$q(\tau, y) := \exp \left\{ -\frac{1}{3m}(\partial_y^{-1}a_8)(\tau, y) \right\}. \quad (2.41)$$

As a consequence, we get

$$\mathcal{L}_3 := \mathcal{M}^{-1}\mathcal{L}_2\mathcal{M} = \partial_\tau + m\partial_{yyy} + a_{12}(\tau, y)\partial_y + a_{13}(\tau, y). \quad (2.42)$$

The proof of the following lemma is straightforward.

Lemma 2.5. *Let $s \geq 0$ and let $a_8 \in C([0, T], H_x^s)$ with $\int_{\mathbb{T}} a_8(\tau, y) dy = 0$ for all $\tau \in [0, T]$. Then for all $\mu \in [0, s]$, the operator \mathcal{M} defined in (2.37), (2.41) and its inverse \mathcal{M}^{-1} belong to $C([0, T], \mathcal{L}(H_x^\mu))$. Note that $\mathcal{M} = \mathcal{M}^T$.*

Furthermore, there exist two positive constants δ_*, σ with the following properties. Assume that $a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10} \in C([0, T], H_x^{s+\sigma})$ and let

$$\delta(\mu) := \|a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10}\|_{T, \mu+\sigma}. \quad (2.43)$$

Then if $\delta(0) \leq \delta_*$, for all $\mu \in [0, s]$ the operator \mathcal{M} and its inverse \mathcal{M}^{-1} satisfy

$$\|\mathcal{M}^{\pm 1} h\|_{T, \mu} \leq C_\mu (\|h\|_{T, \mu} + \delta(\mu) \|h\|_{T, 0}) \quad \forall h \in C([0, T], H_x^\mu), \quad (2.44)$$

for some positive C_μ depending on μ . Moreover, the functions $a_{12}(\tau, y), a_{13}(\tau, y), q(\tau, y)$ defined in (2.39), (2.41) satisfy

$$\|q - 1, a_{12}, \partial_t a_{12}, a_{13}\|_{T, \mu} \leq C_\mu \delta(\mu). \quad (2.45)$$

2.4 Step 4. Translation of the space variable

We consider the change of the space variable $z = y + p(\tau)$ and the operators

$$\mathcal{T}h(\tau, y) := h(\tau, y + p(\tau)), \quad \mathcal{T}^{-1}v(\tau, z) := v(\tau, z - p(\tau)) \quad (2.46)$$

where p is a function $p : [0, T] \rightarrow \mathbb{R}$. The differential operators become $\mathcal{T}^{-1}\partial_y \mathcal{T} = \partial_z$ and $\mathcal{T}^{-1}\partial_\tau \mathcal{T} = \partial_\tau + \{\partial_\tau p(\vartheta)\} \partial_z$. This is a special, simple case of the transformation \mathcal{A} of section 2.1. Thus

$$\mathcal{L}_4 := \mathcal{T}^{-1} \mathcal{L}_3 \mathcal{T} = \partial_\tau + m \partial_{zzz} + a_{14}(\tau, z) \partial_z + a_{15}(\tau, z) \quad (2.47)$$

where

$$a_{14}(\tau, z) := p'(\tau) + (\mathcal{T}^{-1} a_{12})(\tau, z), \quad a_{15}(\tau, z) := (\mathcal{T}^{-1} a_{13})(\tau, z). \quad (2.48)$$

Now we look for $p(\tau)$ such that a_{14} has zero space average. We fix

$$p(\tau) := -\frac{1}{2\pi} \int_0^\tau \int_{\mathbb{T}} a_{12}(s, y) dy ds. \quad (2.49)$$

With this choice of p , after renaming the space-time variables $z = x$ and $\tau = t$, we have

$$\mathcal{L}_4 = \partial_t + m \partial_{xxx} + a_{14}(t, x) \partial_x + a_{15}(t, x), \quad \int_{\mathbb{T}} a_{14}(t, x) dx = 0 \quad \forall t \in [0, T]. \quad (2.50)$$

With direct calculations we prove the following estimates.

Lemma 2.6. *Let $a_{12} \in C([0, T], L_x^2)$. Then the operator \mathcal{T} defined in (2.46), (2.49) belongs to $C([0, T], \mathcal{L}(H_x^s))$ for all $s \in [0, +\infty)$. In fact \mathcal{T} is an isometry, namely*

$$\|\mathcal{T}h\|_{T, s} = \|h\|_{T, s} \quad \forall h \in C([0, T], H_x^s). \quad (2.51)$$

Moreover, \mathcal{T} is invertible and its transpose is $\mathcal{T}^T = \mathcal{T}^{-1}$.

Let $s \geq 0$, and let $a_{12}, \partial_t a_{12}, a_{13} \in C([0, T], H_x^{s+1})$ with $\|a_{12}\|_{T, 0} \leq 1$. Then the functions a_{14}, a_{15}, p defined in (2.48), (2.49) satisfy

$$\sup_{t \in [0, T]} |p(t)| + \|a_{14}, \partial_t a_{14}, a_{15}\|_{T, s} \leq C \|a_{12}, \partial_t a_{12}, a_{13}\|_{T, s+1} \quad (2.52)$$

where C is independent of s .

2.5 Step 5. Elimination of the order one

The goal of this section is to eliminate the term $a_{14}(t, x)\partial_x$. Consider an operator \mathcal{S} of the form

$$\mathcal{S}h := h + \gamma(t, x)\partial_x^{-1}h \quad (2.53)$$

where $\gamma(t, x)$ is a function to be determined. Note that $\partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = \pi_0$ where $\pi_0 h := h - \frac{1}{2\pi} \int_{\mathbb{T}} h dx$. We directly calculate

$$\mathcal{L}_4\mathcal{S} - \mathcal{S}(\partial_t + m\partial_{xxx}) = a_{16}\partial_x + a_{17} + a_{18}\partial_x^{-1} \quad (2.54)$$

where

$$\begin{aligned} a_{16} &:= 3m\gamma_x + a_{14}, & a_{17} &:= a_{15} + (3m\gamma_{xx} + a_{14}\gamma)\pi_0, \\ a_{18} &:= \gamma_t + m\gamma_{xxx} + a_{14}\gamma_x + a_{15}\gamma. \end{aligned} \quad (2.55)$$

We fix γ as

$$\gamma := -\frac{1}{3m}\partial_x^{-1}a_{14}, \quad (2.56)$$

so that $a_{16} = 0$. By the following Lemma 2.7, \mathcal{S} is invertible, and we obtain

$$\mathcal{L}_5 := \mathcal{S}^{-1}\mathcal{L}_4\mathcal{S} = \partial_t + m\partial_{xxx} + \mathcal{R}, \quad \mathcal{R} := \mathcal{S}^{-1}(a_{17} + a_{18}\partial_x^{-1}). \quad (2.57)$$

Lemma 2.7. *There exist positive constants σ, δ_* with the following properties. Let $s \geq 0$, let $a_{14}(t, x), a_{15}(t, x)$ be two functions with $a_{14}, \partial_t a_{14}, a_{15} \in C([0, T], H_x^{s+\sigma})$ and $\int_{\mathbb{T}} a_{14}(t, x) dx = 0$. Let*

$$\delta(\mu) := \|a_{14}, \partial_t a_{14}, a_{15}\|_{T, \mu+\sigma} \quad \forall \mu \in [0, s]. \quad (2.58)$$

If $\delta(0) \leq \delta_$, then the operator \mathcal{S} defined in (2.53), (2.56) belongs to $C([0, T], \mathcal{L}(H_x^\mu))$ for all $\mu \in [0, s]$ and satisfies*

$$\|\mathcal{S}h\|_{T, \mu} \leq C_\mu(\|h\|_{T, \mu} + \delta(\mu)\|h\|_{T, 0}) \quad \forall h \in C([0, T], H_x^\mu), \quad (2.59)$$

for some positive C_μ depending on μ . The operator \mathcal{S} is invertible, and its inverse \mathcal{S}^{-1} , its transpose \mathcal{S}^T and its inverse transpose \mathcal{S}^{-T} all satisfy the same estimate (2.59) as \mathcal{S} .

The operator \mathcal{R} defined in (2.57) belongs to $C([0, T], \mathcal{L}(H_x^\mu))$ for all $\mu \in [0, s]$ and it satisfies

$$\|\mathcal{R}h\|_{T, \mu} \leq C_\mu(\delta(0)\|h\|_{T, \mu} + \delta(\mu)\|h\|_{T, 0}) \quad \forall h \in C([0, T], H_x^\mu). \quad (2.60)$$

The transpose \mathcal{R}^T belongs to $C([0, T], \mathcal{L}(H_x^\mu))$ and satisfies the same estimate (2.60) as \mathcal{R} .

Proof. Estimate $\|\gamma\partial_x^{-1}h\|_{T, \mu}$ by the usual tame estimates for the product of two functions, then use Neumann series in its tame version. \square

3 Observability

In this section we prove the observability of linear operators of the form (2.12). Such observability property will be used in Section 4 in order to prove controllability of the linearized problem. We split the proof into several simple lemmas, starting with a direct consequence of Ingham inequality. Since we actually need observability of a Cauchy problem flowing backwards in time (see Lemma 4.2) with datum at time T , we will accordingly state our lemmas.

Lemma 3.1 (Ingham inequality for $\partial_t + m\partial_{xxx}$). *For every $T > 0$ there exists a positive constant $C_1(T)$ such that, for all $(w_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{C})$, all $m \geq 1/2$,*

$$\int_0^T \left| \sum_{n \in \mathbb{Z}} w_n e^{imn^3 t} \right|^2 dt \geq C_1(T) \sum_{n \in \mathbb{Z}} |w_n|^2.$$

Proof. See e.g. [25], [37]. \square

Lemma 3.2 (Observability for $\partial_t + m\partial_{xxx}$). *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let $v_T \in L^2(\mathbb{T})$, $m \geq 1/2$, and let v satisfy*

$$\partial_t v + m\partial_{xxx} v = 0, \quad v(T) = v_T. \quad (3.1)$$

Then

$$\int_0^T \int_{\omega} |v(t, x)|^2 dx dt \geq C_2 \|v_T\|_{L_x^2}^2 \quad (3.2)$$

with $C_2 := C_1(T)|\omega|$, where $C_1(T)$ is the constant of Proposition 3.1, and $|\omega|$ is the Lebesgue measure of ω .

Proof. Let $v_T(x) = \sum_{n \in \mathbb{Z}} a_n e^{inx}$, so that $v(t, x) = \sum_{n \in \mathbb{Z}} w_n(x) e^{imn^3 t}$ where $w_n(x) := a_n e^{i(n^3 - mn^3 T)}$. By Lemma 3.1, for each $x \in \mathbb{T}$ we have

$$\int_0^T \left| \sum_{n \in \mathbb{Z}} w_n(x) e^{imn^3 t} \right|^2 dt \geq C_1(T) \sum_{n \in \mathbb{Z}} |w_n(x)|^2 = C_1(T) \sum_{n \in \mathbb{Z}} |a_n|^2 = C_1(T) \|v_T\|_{L^2(\mathbb{T})}^2,$$

then we integrate over $x \in \omega$. \square

Lemma 3.3 (Observability of $\mathcal{L}_5 := \partial_t + m\partial_{xxx} + \mathcal{R}$). *Let $T > 0$, let $\omega \subset \mathbb{T}$ be an open set and let $m \geq 1/2$. Let $\mathcal{R} \in C([0, T], \mathcal{L}(L_x^2))$, with $\|\mathcal{R}(t)h\|_0 \leq r_0 \|h\|_0$ for all $h \in L_x^2$, all $t \in [0, T]$, where r_0 is a positive constant. Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0, T], L_x^2)$ be the solution of the Cauchy problem*

$$\partial_t v + m\partial_{xxx} v + \mathcal{R}v = 0, \quad v(T) = v_T, \quad (3.3)$$

which is globally wellposed by Lemma A.2(iii). Then

$$\int_0^T \int_{\omega} |v(t, x)|^2 dx dt \geq C_3 \|v_T\|_{L_x^2}^2$$

with $C_3 := C_2/4$, provided that r_0 is small enough (more precisely, r_0 smaller than a constant depending only on T, C_2 where C_2 is the constant in Lemma 3.2).

Proof. Let v_1 be the solution of $\partial_t v_1 + m\partial_{xxx} v_1 = 0$, $v_1(T) = v_T$, and let $v_2 := v - v_1$. Then v_2 solves

$$(\partial_t + m\partial_{xxx} + \mathcal{R})v_2 = -\mathcal{R}v_1, \quad v_2(T) = 0. \quad (3.4)$$

By (A.10), applied for $s = 0$, $\alpha = 0$, $f = -\mathcal{R}v_1$, we get

$$\|v_2\|_{T,0} \leq 2^{4Tr_0} 4T \|\mathcal{R}v_1\|_{T,0} \leq 2^{4Tr_0} 4Tr_0 \|v_T\|_0. \quad (3.5)$$

Using the elementary inequality $(a + b)^2 \geq \frac{1}{2}a^2 - b^2$ for all $a, b \in \mathbb{R}$,

$$\int_0^T \int_{\omega} |v|^2 dx dt \geq \frac{1}{2} \int_0^T \int_{\omega} |v_1|^2 dx dt - \int_0^T \int_{\omega} |v_2|^2 dx dt.$$

The integral of $|v_1|^2$ is estimated from below by (3.2). The integral of $|v_2|^2$ is bounded by $T\|v_2\|_{T,0}^2$, then use (3.5). \square

Lemma 3.4 (Observability of $\mathcal{L}_4 := \partial_t + m\partial_{xxx} + a_{14}(t, x)\partial_x + a_{15}(t, x)$, a_{14} with zero mean). *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let $m \geq 1/2$ and let $a_{14}(t, x)$, $a_{15}(t, x)$ be two functions, with $a_{14}, \partial_t a_{14}, a_{15} \in C([0, T], H_x^\sigma)$,*

$$\int_{\mathbb{T}} a_{14}(t, x) dx = 0 \quad \forall t \in [0, T], \quad \|a_{14}, \partial_t a_{14}, a_{15}\|_{T, \sigma} \leq \delta. \quad (3.6)$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0, T], L_x^2)$ be the solution of the Cauchy problem

$$\mathcal{L}_4 v = 0, \quad v(T) = v_T, \quad (3.7)$$

which is globally wellposed by Lemma A.3. Then

$$\int_0^T \int_{\omega} |v(t, x)|^2 dx dt \geq C_4 \|v_T\|_{L_x^2}^2$$

with $C_4 := C_3/16$, provided that δ is small enough (more precisely, δ smaller than a constant depending only on T, C_3).

Proof. Following the procedure of Section 2.5, we consider the transformation \mathcal{S} in (2.53), (2.56), which conjugates \mathcal{L}_4 to

$$\mathcal{L}_5 := \mathcal{S}^{-1} \mathcal{L}_4 \mathcal{S} = \partial_t + m\partial_{xxx} + \mathcal{R},$$

where the operator \mathcal{R} is defined in (2.57), (2.55), it belongs to $C([0, T], \mathcal{L}(L_x^2))$, and satisfies the bounds in Lemma 2.7. Let v be the solution of (3.7), and define $\tilde{v} := \mathcal{S}^{-1}v$. Then \tilde{v} solves $\mathcal{L}_5 \tilde{v} = 0$, $\tilde{v}(T) = \tilde{v}_T$ where $\tilde{v}_T := \mathcal{S}^{-1}(T)v_T$, and therefore Lemma 3.3 applies to \tilde{v} if δ is sufficiently small. By Lemmas 2.7, A.3 and Remark A.8 we get

$$\int_0^T \int_{\omega} |(\mathcal{S}^{-1} - I)v|^2 dx dt \leq T \|(\mathcal{S}^{-1} - I)v\|_{T, 0}^2 \leq C\delta^2 \|v\|_{T, 0}^2 \leq C'\delta^2 \|v_T\|_0^2$$

for some constant C' depending on T . We split $\tilde{v} = v + (\mathcal{S}^{-1} - I)v$, and we get

$$\int_0^T \int_{\omega} |\tilde{v}|^2 dx dt \leq 2 \int_0^T \int_{\omega} |v|^2 dx dt + 2C'\delta^2 \|v_T\|_0^2.$$

Moreover $\|v_T\|_0 = \|\mathcal{S}(T)v_T\|_0 \leq 2\|\tilde{v}_T\|_0$, and the thesis follows for δ small enough. \square

Lemma 3.5 (Observability of $\mathcal{L}_3 := \partial_t + m\partial_{xxx} + a_{12}(t, x)\partial_x + a_{13}(t, x)$). *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set and let $m \geq 1/2$. Let $a_{12}(t, x)$, $a_{13}(t, x)$ be two functions, with $a_{12}, \partial_t a_{12}, a_{13} \in C([0, T], H_x^\sigma)$,*

$$\|a_{12}, \partial_t a_{12}, a_{13}\|_{T, \sigma} \leq \delta. \quad (3.8)$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0, T], L_x^2)$ be the solution of the Cauchy problem

$$\mathcal{L}_3 v = 0, \quad v(T) = v_T, \quad (3.9)$$

which is globally wellposed by Lemma A.4. Then

$$\int_0^T \int_{\omega} |v(t, x)|^2 dx dt \geq C_5 \|v_T\|_{L_x^2}^2 \quad (3.10)$$

for some $C_5 > 0$ depending on T, ω , provided that δ in (3.8) is sufficiently small (more precisely, δ smaller than a constant depending on T, ω, C_4).

Proof. Following the procedure of Section 2.4, we consider the transformation \mathcal{T} defined in (2.46), (2.49), which conjugates \mathcal{L}_3 to

$$\mathcal{L}_4 := \mathcal{T}^{-1} \mathcal{L}_3 \mathcal{T} = \partial_t + m \partial_{xxx} + a_{14}(t, x) \partial_x + a_{15}(t, x),$$

where a_{14}, a_{15} are defined in (2.48), and $\int_{\mathbb{T}} a_{14}(t, x) dx = 0$. By (2.52), the function p defined in (2.49) satisfies $|p(t)| \leq C\delta$ for all $t \in [0, T]$. Let v be the solution of the Cauchy problem (3.9). Then $\tilde{v} := \mathcal{T}^{-1}v$ solves $\mathcal{L}_4 \tilde{v} = 0$, $\tilde{v}(T) = \mathcal{T}^{-1}(T)v_T$. Let $\omega_1 = [\alpha_1, \beta_1]$ be an interval contained in ω . For δ small enough, one has

$$[\alpha_1 - p(t), \beta_1 - p(t)] \subseteq [\alpha_1 - \delta, \beta_1 + \delta] \subset \omega \quad \forall t \in [0, T].$$

The change of variable $x - p(t) = y$, $dx = dy$ gives

$$\int_0^T \int_{\omega_1} |\tilde{v}(t, x)|^2 dx dt = \int_0^T \int_{\alpha_1 - p(t)}^{\beta_1 - p(t)} |v(t, y)|^2 dy dt \leq \int_0^T \int_{\omega} |v(t, y)|^2 dy dt.$$

By (2.52), for δ small enough, Lemma 3.4 can be applied to \tilde{v} on the interval ω_1 and the thesis follows, since $\|\tilde{v}(T)\|_0 = \|\mathcal{T}^{-1}(T)v_T\|_0 = \|v_T\|_0$. \square

Lemma 3.6 (Observability of $\mathcal{L}_2 := \partial_t + m \partial_{xxx} + a_8(t, x) \partial_{xx} + a_9(t, x) \partial_x + a_{10}(t, x)$). *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set and let $m \geq 1/2$. Let $a_8(t, x)$, $a_9(t, x)$, $a_{10}(t, x)$ be three functions, with $a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10} \in C([0, T], H_x^\sigma)$,*

$$\int_{\mathbb{T}} a_8(t, x) dx = 0 \quad \forall t \in [0, T], \quad \|a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10}\|_{T, \sigma} \leq \delta. \quad (3.11)$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0, T], L_x^2)$ be the solution of the Cauchy problem

$$\mathcal{L}_2 v = 0, \quad v(T) = v_T, \quad (3.12)$$

which is globally wellposed by Lemma A.5. Then

$$\int_0^T \int_{\omega} |v(t, x)|^2 dx dt \geq C_6 \|v_T\|_{L_x^2}^2 \quad (3.13)$$

for some $C_6 > 0$ depending on T, ω , provided that δ in (3.11) is sufficiently small (more precisely, δ smaller than a constant depending on T, ω, C_5).

Proof. Following the procedure of Section 2.3, we consider the multiplication operator \mathcal{M} defined in (2.37), (2.41), which conjugates \mathcal{L}_2 to

$$\mathcal{M}^{-1} \mathcal{L}_2 \mathcal{M} = \mathcal{L}_3, \quad \mathcal{L}_3 = \partial_t + m \partial_{xxx} + a_{12}(t, x) \partial_x + a_{13}(t, x),$$

where a_{12}, a_{13} are defined in (2.39). Let v be the solution of the Cauchy problem (3.12). Then $\tilde{v} := \mathcal{M}^{-1}v$ solves $\mathcal{L}_3 \tilde{v} = 0$, $\tilde{v}(T) = \mathcal{M}^{-1}(T)v_T$. Using (2.45), we have

$$\int_0^T \int_{\omega} |v(t, x)|^2 dx dt = \int_0^T \int_{\omega} |\tilde{v}|^2 dx dt + \int_0^T \int_{\omega} |\tilde{v}|^2 (|q|^2 - 1) dx dt \geq (C_5 - C\delta) \|v_T\|_0^2.$$

The first of the two integrals has been estimated from below by applying Lemma 3.5 to \mathcal{L}_3 (by Lemma 2.5, this can be done provided that δ is sufficiently small). The second integral has been estimated using the bound (2.45), since $|q(t) - 1| \leq C\|q - 1\|_{T, 1} \leq C'\delta$. Moreover, we have used the inequality $\|\tilde{v}\|_{T, 0} \leq C\|\tilde{v}_T\|_0$ from Lemma A.4. The thesis follows with $C_6 := C_5/2$ by choosing δ small enough. \square

Lemma 3.7 (Observability of $\mathcal{L}_1 := \partial_t + a_4(t)\partial_{xxx} + a_5(t, x)\partial_{xx} + a_6(t, x)\partial_x + a_7(t, x)$).

There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let a_4, a_5, a_6, a_7 be four functions, with $a_4 \in C^1([0, T], \mathbb{R})$, $a_5, \partial_t a_5, a_6, \partial_t a_6, a_7 \in C([0, T], H_x^\sigma)$, satisfying

$$\int_{\mathbb{T}} a_5(t, x) dx = 0 \quad \forall t \in [0, T], \quad \|a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T, \sigma} + |a_4 - 1, a_4'|_T \leq \delta. \quad (3.14)$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0, T], L_x^2)$ be the solution of the Cauchy problem

$$\mathcal{L}_1 v = 0, \quad v(T) = v_T, \quad (3.15)$$

which is globally wellposed by Lemma A.6. Then

$$\int_0^T \int_{\omega} |v(t, x)|^2 dx dt \geq C_7 \|v_T\|_{L_x^2}^2 \quad (3.16)$$

for some $C_7 > 0$ depending on T, ω , provided that δ in (3.14) is sufficiently small (more precisely, δ smaller than a constant depending on T, ω, C_6).

Proof. Following the procedure of Section 2.2, we consider the re-parametrization of time \mathcal{B} defined in (2.25), (2.30), which conjugates \mathcal{L}_1 to

$$\mathcal{B}^{-1} \mathcal{L}_1 \mathcal{B} = \rho \mathcal{L}_2, \quad \mathcal{L}_2 = \partial_\tau + m \partial_{xxx} + a_8(\tau, x) \partial_{xx} + a_9(\tau, x) \partial_x + a_{10}(\tau, x),$$

where ρ, a_8, a_9, a_{10} are defined in (2.28), (2.32) and $\int_{\mathbb{T}} a_8(\tau, x) dx = 0$ for all $\tau \in [0, T]$. Let v be the solution of the Cauchy problem (3.15). Then $\tilde{v} := \mathcal{B}^{-1} v$ solves $\mathcal{L}_2 \tilde{v} = 0$, $\tilde{v}(T) = \mathcal{B}^{-1}(T) v_T$. Using (2.35), we have

$$\begin{aligned} \int_0^T \int_{\omega} |v(t, x)|^2 dx dt &= \int_0^T \int_{\omega} |\tilde{v}(\psi(t), x)|^2 dx dt \\ &= \int_0^T \int_{\omega} |\tilde{v}(\psi(t), x)|^2 [\psi'(t) + (1 - \psi'(t))] dx dt \\ &= \int_0^T \int_{\omega} |\tilde{v}(\tau, x)|^2 dx d\tau + \int_0^T \int_{\omega} |\tilde{v}(\psi(t), x)|^2 (1 - \psi'(t)) dx dt \\ &\geq (C_6 - C\delta) \|v_T\|_0^2. \end{aligned}$$

The first of the two integrals has been estimated from below by applying Lemma 3.6 to \mathcal{L}_2 (by Lemma 2.4, this can be done provided that δ is sufficiently small). The second integral has been estimated using the bound (2.35) for $|\psi'(t) - 1|$ and also the inequality $\|\tilde{v}\|_{T,0} \leq C \|\tilde{v}_T\|_0$ from Lemma A.5. The thesis follows with $C_7 := C_6/2$ by choosing δ small enough, since $\|\tilde{v}_T\|_0 = \|\mathcal{B}^{-1}(T) v_T\|_0 = \|v_T\|_0$. \square

Lemma 3.8 (Observability of $\mathcal{L}_0 := \partial_t + (1 + a_3)\partial_{xxx} + a_2\partial_{xx} + a_1\partial_x + a_0$). There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let $c \in \mathbb{R}$ and $a_3(t, x), a_2(t, x), a_1(t, x), a_0(t, x)$ be four functions with $a_2 = c\partial_x a_3$,

$$\|\partial_{tt} a_3, \partial_t a_3, a_3, \partial_t a_1, a_1, a_0\|_{T, \sigma} \leq \delta. \quad (3.17)$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0, T], L_x^2)$ be the solution of the Cauchy problem

$$\mathcal{L}_0 v = 0, \quad v(T) = v_T, \quad (3.18)$$

which is globally wellposed by Lemma A.7. Then

$$\int_0^T \int_{\omega} |v(t, x)|^2 dx dt \geq C_8 \|v_T\|_{L_x^2}^2 \quad (3.19)$$

for some $C_8 > 0$ depending on T, ω , provided that δ in (3.17) is sufficiently small (more precisely, δ smaller than a constant depending on T, ω, C_7).

Proof. Following the procedure of Section 2.1, we consider the transformation \mathcal{A} defined in (2.9), (2.16), (2.17), (2.18), which conjugates \mathcal{L}_0 to

$$\mathcal{A}^{-1} \mathcal{L}_0 \mathcal{A} = \mathcal{L}_1 = \partial_t + a_4(t) \partial_{xxx} + a_5(t, x) \partial_{xx} + a_6(t, x) \partial_x + a_7(t, x)$$

(see (2.19)), where a_4, a_5, a_6, a_7 are defined in (2.14) and $\int_{\mathbb{T}} a_5(t, x) = 0$ for all $t \in [0, T]$. Let v be the solution of the Cauchy problem (3.18). Then $\tilde{v} := \mathcal{A}^{-1} v$ solves $\mathcal{L}_1 \tilde{v} = 0$, $\tilde{v}(T) = \tilde{v}_T$, where $\tilde{v}_0 := \mathcal{A}^{-1}(0) v_0$. Let $\omega_1 = [\alpha_1, \beta_1] \subset \omega$. By (2.22) in Lemma 2.3, for δ sufficiently small Lemma 3.7 applies to \tilde{v} on ω_1 , and

$$\int_0^T \int_{\omega_1} |\tilde{v}|^2 dy dt \geq C_7 \|\tilde{v}_T\|_0^2.$$

By Lemma 2.3, $\|v_T\|_0 = \|\mathcal{A}(T) \tilde{v}_T\|_0 \leq C \|\tilde{v}_T\|_0$. The change of integration variable $y = x + \beta(t, x)$, $dy = (1 + \beta_x(t, x)) dx$ gives

$$\begin{aligned} \int_0^T \int_{\omega_1} |\tilde{v}|^2 dy dt &= \int_0^T \int_{\omega_1} |(\mathcal{A}^{-1} v)(t, y)|^2 dy dt \\ &= \int_0^T \int_{\omega_2(t)} \frac{|v(t, x)|^2}{1 + \beta_x(t, x)} dx dt \leq 2 \int_0^T \int_{\omega} |v(t, x)|^2 dx dt, \end{aligned}$$

where $\omega_2(t) := \{x : x + \beta(t, x) \in \omega_1\}$. We have used the fact that, for δ small enough, $\omega_2(t) \subset \omega$, and the bound (2.22) for $|\beta_x(t, x)| \leq C \|\beta\|_{T,2} \leq C' \delta$. \square

4 Controllability

In this section we prove the controllability of the linearized operator \mathcal{L}_0 , using its observability (Lemma 3.8), by means of the HUM method. We also prove higher regularity of the control.

Lemma 4.1 (Controllability of \mathcal{L}_0). *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let a_3, a_2, a_1, a_0 be four functions of (t, x) with $a_2 = 2\partial_x a_3$ satisfying (3.17). Let \mathcal{L}_0 be the linear operator*

$$\mathcal{L}_0 := \partial_t + (1 + a_3) \partial_{xxx} + a_2 \partial_{xx} + a_1 \partial_x + a_0. \quad (4.1)$$

(i) Existence. *There exist constants δ_0, C such that, if δ in (3.17) is smaller than δ_0 , then the following property holds. Given any three functions $g_1(t, x)$, $g_2(x)$, $g_3(x)$, with $g_1 \in C([0, T], L_x^2)$, $g_2, g_3 \in L_x^2$, there exists a function $\varphi \in C([0, T], L_x^2)$ such that the solution h of the Cauchy problem*

$$\mathcal{L}_0 h = g_1 + \chi_{\omega} \varphi, \quad h(0) = g_2 \quad (4.2)$$

satisfies $h(T) = g_3$. (Note that the Cauchy problem (4.2) is globally well-posed by Lemma A.7). Moreover

$$\|\varphi\|_{T,0} \leq C(\|g_1\|_{T,0} + \|g_2\|_0 + \|g_3\|_0). \quad (4.3)$$

(ii) Uniqueness. Let \mathcal{L}_0^* be the linear operator

$$\mathcal{L}_0^* \psi := -\partial_t \psi - \partial_{xxx} \{(1 + a_3)\psi\} + \partial_{xx}(a_2 \psi) - \partial_x(a_1 \psi) + a_0 \psi. \quad (4.4)$$

The control φ in (i) is the unique solution of the equation $\mathcal{L}_0^* \varphi = 0$ such that the solution h of the Cauchy problem (4.2) satisfies $h(T) = g_3$.

The proof of Lemma 4.1 is given below, and it is based on the following classical lemma. In this section we use the standard notation $\langle u, v \rangle := \int_{\mathbb{T}} uv \, dx$.

Lemma 4.2. Let a_3, a_2, a_1, a_0 be functions satisfying (3.17) and $a_2 = 2\partial_x a_3$. Let \mathcal{L}_0^* be the operator defined in (4.4). For every (g_1, g_2, g_3) with $g_1 \in C([0, T], L_x^2)$, $g_2, g_3 \in L_x^2$ there exists a unique $\varphi_1 \in L_x^2$ such that for all $\psi_1 \in L_x^2$, the solutions $\varphi, \psi \in C([0, T], L_x^2)$ of the Cauchy problems

$$\begin{cases} \mathcal{L}_0^* \varphi = 0 \\ \varphi(T) = \varphi_1 \end{cases} \quad \begin{cases} \mathcal{L}_0^* \psi = 0 \\ \psi(T) = \psi_1 \end{cases} \quad (4.5)$$

satisfy

$$\int_0^T \langle g_1 + \chi_\omega \varphi, \psi \rangle \, dt + \langle g_2, \psi(0) \rangle - \langle g_3, \psi(T) \rangle = 0 \quad (4.6)$$

(note that the global well-posedness of the Cauchy problems (4.5) follows from Lemma A.7 and Remark A.8). Moreover φ satisfies (4.3).

Proof. Given $\varphi_1, \psi_1 \in L_x^2$, let φ, ψ be the solutions of the Cauchy problems (4.5), and define

$$B(\varphi_1, \psi_1) := \int_0^T \langle \chi_\omega \varphi, \psi \rangle \, dt, \quad \Lambda(\psi_1) := \langle g_3, \psi(T) \rangle - \langle g_2, \psi(0) \rangle - \int_0^T \langle g_1, \psi \rangle \, dt. \quad (4.7)$$

The bilinear map $B : L_x^2 \times L_x^2 \rightarrow \mathbb{R}$ is well defined and continuous because $|\chi_\omega(x)| \leq 1$ and, by Lemma A.7 and Remark A.8, $\|\varphi\|_{T,0} \leq C\|\varphi_1\|_0$, and similarly for ψ . Moreover B is coercive by Lemma 3.8 and Remark 2.2. The linear functional Λ is bounded, with

$$|\Lambda(\psi_1)| \leq C\|g\|_{T,0}\|\psi_1\|_0 \quad \forall \psi_1 \in L_x^2, \quad \|g\|_{T,0} := \|g_1\|_{T,0} + \|g_2\|_0 + \|g_3\|_0.$$

Thus, by Riesz representation theorem (or Lax-Milgram), there exists a unique $\varphi_1 \in L_x^2$ such that

$$B(\varphi_1, \psi_1) = \Lambda(\psi_1) \quad \forall \psi_1 \in L_x^2. \quad (4.8)$$

Moreover $\|\varphi_1\|_0 \leq C\|\Lambda\|_{\mathcal{L}(L_x^2, \mathbb{R})} \leq C'\|g\|_{T,0}$. Since $\|\varphi\|_{T,0} \leq C\|\varphi_1\|_0$, we get (4.3). \square

Proof of Lemma 4.1. (i). Let $\varphi_1 \in L_x^2$ be the unique solution of (4.8) given by Lemma 4.2. Consider any $\psi_1 \in L_x^2$, and let $\varphi, \psi \in C([0, T], L_x^2)$ be the unique solutions of the

Cauchy problems (4.5). Recalling (4.6), (4.2) and integrating by parts, we have

$$\begin{aligned}
0 &= \int_0^T \langle g_1 + \chi_\omega \varphi, \psi \rangle dt + \langle g_2, \psi(0) \rangle - \langle g_3, \psi(T) \rangle \\
&= \int_0^T \langle \mathcal{L}_0 h, \psi \rangle dt + \langle g_2, \psi(0) \rangle - \langle g_3, \psi(T) \rangle \\
&= \langle h(T), \psi(T) \rangle - \langle h(0), \psi(0) \rangle + \int_0^T \langle h, \mathcal{L}_0^* \psi \rangle dt + \langle g_2, \psi(0) \rangle - \langle g_3, \psi(T) \rangle \\
&= \langle h(T), \psi(T) \rangle - \langle g_3, \psi(T) \rangle \\
&= \langle h(T) - g_3, \psi_1 \rangle,
\end{aligned}$$

from which it follows that $h(T) = g_3$.

(ii). Assume that $\tilde{\varphi} \in C([0, T], L_x^2)$ satisfies $\mathcal{L}_0^* \tilde{\varphi} = 0$ and it has the property that the solution h of the Cauchy problem (4.2) satisfies $h(T) = g_3$. Let $\tilde{\varphi}_1 := \tilde{\varphi}(T)$. The same integration by parts as above shows that $B(\tilde{\varphi}_1, \psi_1) = \Lambda(\psi_1)$ for all $\psi_1 \in L_x^2$. By the uniqueness in Lemma 4.2, $\tilde{\varphi}_1 = \varphi_1$. \square

Lemma 4.3 (Higher regularity). *Let $T, \omega, a_3, a_2, a_1, a_0, \mathcal{L}_0, g_1, g_2, g_3$ be as in Lemma 4.1. There exist two positive constants δ_*, σ with the following property. Let $s > 0$ be given. Assume that $a_0, a_1, a_2, a_3 \in C^2([0, T], H_x^{s+\sigma})$. Let*

$$\delta(\mu) := \sum_{k=0,1,2, \ i=0,1,2,3} \|\partial_t^k a_i\|_{T, \mu+\sigma}, \quad \mu \in [0, s].$$

Let $\|g\|_{T,s} := \|g_1\|_{T,s} + \|g_2\|_s + \|g_3\|_s < \infty$. If $\delta(0) \leq \delta_$, then the control φ constructed in Lemma 4.1 and the solution h of (4.2) satisfy*

$$\|\varphi, h\|_{T,s} \leq C_s (\|g\|_{T,s} + \delta(s) \|g\|_{T,0}) \quad (4.9)$$

for some positive C_s depending on s, T, ω . Moreover, if $g_1 \in C^1([0, T], H_x^s)$, then

$$\|\partial_t \varphi, \partial_t h\|_{T,s+3} + \|\partial_{tt} \varphi, \partial_{tt} h\|_{T,s} \leq C_s \{\|g\|_{T,s+6} + \|\partial_t g_1\|_{T,s} + \delta(s) \|g\|_{T,6}\}. \quad (4.10)$$

Proof. Let $g_1 \in C([0, T], H_x^s)$, $g_2, g_3 \in H_x^s$. Let $\varphi, h \in C([0, T], L_x^2)$ be the solution of the control problem constructed in Lemma 4.1, namely

$$\mathcal{L}_0^* \varphi = 0, \quad \mathcal{L}_0 h = \chi_\omega \varphi + g_1, \quad h(0) = g_2, \quad h(T) = g_3. \quad (4.11)$$

To prove that $h, \varphi \in C([0, T], H_x^s)$, it is convenient to use the transformations of Section 2, to prove higher regularity for the solution $\tilde{h}, \tilde{\varphi}$ of the transformed control problem, and then to go back to h, φ proving their higher regularity. Recall that

$$\mathcal{L}_0 = \mathcal{A} \mathcal{B} \rho \mathcal{M} \mathcal{T} \mathcal{S} \mathcal{L}_5 \mathcal{S}^{-1} \mathcal{T}^{-1} \mathcal{M}^{-1} \mathcal{B}^{-1} \mathcal{A}^{-1}, \quad (4.12)$$

where $\mathcal{L}_5 = \partial_t + m \partial_{xxx} + \mathcal{R}$ and $\mathcal{A}, \mathcal{B}, \rho, \mathcal{M}, \mathcal{T}, \mathcal{S}$ are defined in Section 2. Define

$$\mathcal{L}_5^* := -\partial_t - m \partial_{xxx} + \mathcal{R}^T, \quad (4.13)$$

where \mathcal{R}^T is the L_x^2 -adjoint of \mathcal{R} . Let

$$\begin{aligned}
\tilde{h} &:= (\mathcal{A} \mathcal{B} \mathcal{M} \mathcal{T} \mathcal{S})^{-1} h, & \tilde{g}_1 &:= (\mathcal{A} \mathcal{B} \rho \mathcal{M} \mathcal{T} \mathcal{S})^{-1} g_1, \\
\tilde{g}_2 &:= (\mathcal{A} \mathcal{B} \mathcal{M} \mathcal{T} \mathcal{S})^{-1}|_{t=0} g_2, & \tilde{g}_3 &:= (\mathcal{A} \mathcal{B} \mathcal{M} \mathcal{T} \mathcal{S})^{-1}|_{t=T} g_3, \\
\tilde{\varphi} &:= \mathcal{S}^T \mathcal{T}^T \mathcal{M}^T \mathcal{B}^{-1} \mathcal{A}^T \varphi, & K \tilde{\varphi} &:= (\mathcal{A} \mathcal{B} \rho \mathcal{M} \mathcal{T} \mathcal{S})^{-1} (\chi_\omega (\mathcal{S}^T \mathcal{T}^T \mathcal{M}^T \mathcal{B}^{-1} \mathcal{A}^T)^{-1} \tilde{\varphi}).
\end{aligned} \quad (4.14)$$

Note that, except for $\mathcal{S}^{-1}, \mathcal{S}^{-T}$, the operator K is a multiplication operator, namely

$$K\tilde{\varphi} = \mathcal{S}^{-1}(\zeta\mathcal{S}^{-T}\tilde{\varphi}), \quad \text{where} \quad \zeta(t, x) := \rho^{-1}\mathcal{T}^{-1}\mathcal{M}^{-2}\mathcal{B}^{-1}\mathcal{A}^{-1}[(1 + \beta_x)\chi_\omega]. \quad (4.15)$$

Since $h, \varphi \in C([0, T], L_x^2)$, and $g_1 \in C([0, T], H_x^s)$, $g_2, g_3 \in H_x^s$, by (4.14) and the estimates for $\mathcal{A}, \mathcal{B}, \rho, \mathcal{M}, \mathcal{T}, \mathcal{S}$ in section 2, one has

$$\tilde{h}, \tilde{\varphi}, K\tilde{\varphi} \in C([0, T], L_x^2), \quad \tilde{g}_1 \in C([0, T], H_x^s), \quad \tilde{g}_2, \tilde{g}_3 \in H_x^s.$$

Since h, φ satisfy (4.11), one proves that $\tilde{h}, \tilde{\varphi}$ satisfy

$$\mathcal{L}_5^*\tilde{\varphi} = 0, \quad \mathcal{L}_5\tilde{h} = K\tilde{\varphi} + \tilde{g}_1, \quad \tilde{h}(0) = \tilde{g}_2, \quad \tilde{h}(T) = \tilde{g}_3. \quad (4.16)$$

The last three equations in (4.16) are straightforward. To prove that $\mathcal{L}_5^*\tilde{\varphi} = 0$, we start from the equality

$$\langle \varphi(T), v(T) \rangle - \langle \varphi(0), v(0) \rangle = \int_0^T \langle \varphi, \mathcal{L}_0 v \rangle dt \quad \forall v \in C^\infty([0, T] \times \mathbb{T})$$

(which is a weak form of $\mathcal{L}_0^*\varphi = 0$), we recall (4.12), and apply all the changes of variables $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{T}, \mathcal{S}$ in the integral. Thus $\tilde{h}, \tilde{\varphi}$ solve this control problem:

$$\begin{cases} \text{Given } \tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \text{ find } \tilde{\varphi} \text{ such that the solution } \tilde{h} \\ \text{of the Cauchy problem } \mathcal{L}_5\tilde{h} = K\tilde{\varphi} + \tilde{g}_1, \tilde{h}(0) = \tilde{g}_2 \\ \text{satisfies } \tilde{h}(T) = \tilde{g}_3, \text{ and moreover } \tilde{\varphi} \text{ solves } \mathcal{L}_5^*\tilde{\varphi} = 0. \end{cases} \quad (4.17)$$

The function $\tilde{\varphi}$ is the unique solution of (4.17). To prove it, assume that $\tilde{\varphi}_{bis} \in C([0, T], L_x^2)$ solves (4.17), and let \tilde{h}_{bis} be the solution of the corresponding Cauchy problem $\mathcal{L}_5\tilde{h}_{bis} = K\tilde{\varphi}_{bis} + \tilde{g}_1$, $\tilde{h}_{bis}(0) = \tilde{g}_2$. Define

$$h_{bis} := \mathcal{A}\mathcal{B}\mathcal{M}\mathcal{T}\mathcal{S}\tilde{h}_{bis}, \quad \varphi_{bis} := \mathcal{A}^{-T}\mathcal{B}\mathcal{M}^{-T}\mathcal{T}^{-T}\mathcal{S}^{-T}\tilde{\varphi}_{bis}.$$

Then h_{bis}, φ_{bis} solve (4.11). By the uniqueness in Lemma 4.1(ii) it follows that $\varphi_{bis} = \varphi$, $h_{bis} = h$. Therefore $\tilde{\varphi}_{bis} = \tilde{\varphi}$ and $\tilde{h}_{bis} = \tilde{h}$.

Now we prove that $\tilde{h}, \tilde{\varphi} \in C([0, T], H_x^s)$. We follow an argument used by Dehman-Lebeau [19, Lemma 4.2], Laurent [32, Lemma 3.1], and [3, Proposition 8.1]. First, we prove the thesis for $\tilde{g}_1 = 0, \tilde{g}_3 = 0$. Consider the map

$$S: L_x^2 \rightarrow L_x^2, \quad S\tilde{\varphi}_1 = \tilde{h}(0) \quad (4.18)$$

obtained by the composition $\tilde{\varphi}_1 \mapsto \tilde{\varphi} \mapsto \tilde{h} \mapsto \tilde{h}(0)$, where $\tilde{\varphi}, \tilde{h}$ are the solutions of the Cauchy problems

$$\begin{cases} \mathcal{L}_5^*\tilde{\varphi} = 0 \\ \tilde{\varphi}(T) = \tilde{\varphi}_1, \end{cases} \quad \begin{cases} \mathcal{L}_5\tilde{h} = K\tilde{\varphi} \\ \tilde{h}(T) = 0. \end{cases} \quad (4.19)$$

From the existence and uniqueness of $\tilde{\varphi}_1 \in L_x^2$ such that $\tilde{\varphi}$ solves (4.17) it follows that S is an isomorphism of L_x^2 . The initial datum \tilde{g}_2 is given, so we fix $\tilde{\varphi}_1 \in L_x^2$ such that $S\tilde{\varphi}_1 = \tilde{g}_2$. We have to estimate $\|\Lambda^s\tilde{\varphi}_1\|_0 \leq C\|S\Lambda^s\tilde{\varphi}_1\|_0$, where Λ^s is the Fourier multiplier of symbol $\langle \xi \rangle^s := (1 + \xi^2)^{s/2}$, $s > 0$. To study the commutator $[S, \Lambda^s]$, we compare $(\Lambda^s\tilde{\varphi}, \Lambda^s\tilde{h})$ with $(\tilde{\varphi}, \tilde{h})$ defined by

$$\begin{cases} \mathcal{L}_5^*\tilde{\varphi} = 0 \\ \tilde{\varphi}(T) = \Lambda^s\varphi_1, \end{cases} \quad \begin{cases} \mathcal{L}_5\tilde{h} = K\tilde{\varphi} \\ \tilde{h}(T) = 0. \end{cases} \quad (4.20)$$

The difference $\Lambda^s \tilde{\varphi} - \bar{\varphi}$ satisfies

$$\begin{cases} \mathcal{L}_5^*(\Lambda^s \tilde{\varphi} - \bar{\varphi}) = \mathcal{F}_1, \\ (\Lambda^s \tilde{\varphi} - \bar{\varphi})(T) = 0 \end{cases} \quad \text{where } \mathcal{F}_1 := [\mathcal{L}_5^*, \Lambda^s] \tilde{\varphi} = [\mathcal{R}^T, \Lambda^s] \tilde{\varphi}. \quad (4.21)$$

From Lemma A.2 and Remark A.8, $\|\Lambda^s \tilde{\varphi} - \bar{\varphi}\|_{T,0} \leq C \|\mathcal{F}_1\|_{T,0}$. We recall the classical estimate for the commutator of Λ^s and any multiplication operator $h \mapsto ah$:

$$\|[\Lambda^s, a]h\|_0 \leq C_s(\|a\|_2 \|h\|_{s-1} + \|a\|_{s+1} \|h\|_0). \quad (4.22)$$

By (4.22) and formulas (2.53), (2.56), (2.57), the commutator $\mathcal{F}_1 = [\mathcal{R}^T, \Lambda^s] \tilde{\varphi}$ satisfies

$$\begin{aligned} \|\mathcal{F}_1\|_{T,0} &\leq C_s(\|a_{14}, a_{17}, a_{18}\|_{T,\sigma} \|\tilde{\varphi}\|_{T,s-1} + \|a_{14}, a_{17}, a_{18}\|_{T,s+\sigma} \|\tilde{\varphi}\|_{T,0}) \\ &\leq C_s(\delta(0) \|\tilde{\varphi}\|_{T,s-1} + \delta(s) \|\tilde{\varphi}\|_{T,0}). \end{aligned} \quad (4.23)$$

The difference $\Lambda^s \tilde{h} - \bar{h}$ satisfies

$$\begin{cases} \mathcal{L}_5(\Lambda^s \tilde{h} - \bar{h}) = K(\Lambda^s \tilde{\varphi} - \bar{\varphi}) + \mathcal{F}_2, \\ (\Lambda^s \tilde{h} - \bar{h})(T) = 0, \end{cases} \quad \text{where } \mathcal{F}_2 := [\mathcal{R}^T, \Lambda^s] \tilde{h} + [\Lambda^s, K] \tilde{\varphi}. \quad (4.24)$$

We have $\|K(\Lambda^s \tilde{\varphi} - \bar{\varphi})\|_{T,0} \leq C \|\Lambda^s \tilde{\varphi} - \bar{\varphi}\|_{T,0} \leq C \|\mathcal{F}_1\|_{T,0}$, and therefore, by Lemma A.2,

$$\|\Lambda^s \tilde{h} - \bar{h}\|_{T,0} \leq C(\|\mathcal{F}_1\|_{T,0} + \|\mathcal{F}_2\|_{T,0}). \quad (4.25)$$

Using (4.22) and (4.15), we get

$$\|\mathcal{F}_2\|_{T,0} \leq C_s(\|\tilde{h}, \tilde{\varphi}\|_{T,s-1} + \delta(s) \|\tilde{h}, \tilde{\varphi}\|_{T,0}). \quad (4.26)$$

By (4.23), (4.25) and (4.26) we deduce that

$$\|\Lambda^s \tilde{h} - \bar{h}\|_{T,0} \leq C_s(\|\tilde{h}, \tilde{\varphi}\|_{T,s-1} + \delta(s) \|\tilde{h}, \tilde{\varphi}\|_{T,0}).$$

By (4.19), Lemma A.2 and Remark A.8,

$$\|\tilde{h}, \tilde{\varphi}\|_{T,\mu} \leq C_\mu(\|\tilde{\varphi}\|_{T,\mu} + \delta(\mu) \|\tilde{\varphi}\|_{T,0}) \leq C_\mu(\|\tilde{\varphi}_1\|_\mu + \delta(\mu) \|\tilde{\varphi}_1\|_0), \quad \mu \geq 0. \quad (4.27)$$

Therefore

$$\|(\Lambda^s \tilde{h} - \bar{h})(0)\|_0 \leq \|\Lambda^s \tilde{h} - \bar{h}\|_{T,0} \leq C_s(\|\tilde{\varphi}_1\|_{s-1} + \delta(s) \|\tilde{\varphi}_1\|_0). \quad (4.28)$$

Since $S\tilde{\varphi}_1 = \tilde{h}(0) = \tilde{g}_2$, we have $\Lambda^s \tilde{h}(0) = \Lambda^s g_2$. Moreover, by the definition of S in (4.18)-(4.19), $\bar{h}(0) = S\Lambda^s \tilde{\varphi}_1$. Thus

$$\|S\Lambda^s \tilde{\varphi}_1\|_0 \leq \|(\Lambda^s \tilde{h} - \bar{h})(0)\|_0 + \|\Lambda^s \tilde{h}(0)\|_0 \leq C_s(\|\tilde{\varphi}_1\|_{s-1} + \delta(s) \|\tilde{\varphi}_1\|_0) + \|\tilde{g}_2\|_s. \quad (4.29)$$

Since S is an isomorphism of L_x^2 , $\|\Lambda^s \tilde{\varphi}_1\|_0 \leq C \|S\Lambda^s \tilde{\varphi}_1\|_0$, whence

$$\|\tilde{\varphi}_1\|_s \leq C_s(\|\tilde{g}_2\|_s + \|\tilde{\varphi}_1\|_{s-1} + \delta(s) \|\tilde{\varphi}_1\|_0). \quad (4.30)$$

Since $\|\tilde{\varphi}_1\|_0 \leq C \|\tilde{g}_2\|_0$, by induction we deduce that

$$\|\tilde{\varphi}_1\|_s \leq C_s(\|\tilde{g}_2\|_s + \delta(s) \|\tilde{g}_2\|_0). \quad (4.31)$$

By (4.27), we obtain

$$\|\tilde{h}, \tilde{\varphi}\|_{T,s} \leq C_s(\|\tilde{g}_2\|_s + \delta(s)\|\tilde{g}_2\|_0), \quad (4.32)$$

which is the thesis in the case $\tilde{g}_1 = 0, \tilde{g}_3 = 0$.

Now we prove the higher regularity of $\tilde{h}, \tilde{\varphi}$ removing the assumption $\tilde{g}_1 = 0, \tilde{g}_3 = 0$. Let $\tilde{g}_1 \in C([0, T], H_x^s)$, $\tilde{g}_2, \tilde{g}_3 \in H_x^s$, and let $\tilde{h}, \tilde{\varphi}$ be the solution of (4.17). Let w be the solution of the problem

$$\mathcal{L}_5 w = \tilde{g}_1, \quad w(T) = \tilde{g}_3.$$

By Lemma A.2, $w \in C([0, T], H_x^s)$, with

$$\|w\|_{T,s} \leq C_s\{\|\tilde{g}_1\|_{T,s} + \|\tilde{g}_3\|_s + \delta(s)(\|\tilde{g}_1\|_{T,0} + \|\tilde{g}_3\|_0)\}. \quad (4.33)$$

Let $v := \tilde{h} - w$. Then

$$\mathcal{L}_5 v = K\tilde{\varphi}, \quad v(0) = \tilde{g}_2 - w(0), \quad v(T) = 0.$$

This means that $v, \tilde{\varphi}$ solve (4.17) where $(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)$ are replaced by $(0, \tilde{g}_2 - w(0), 0)$. Hence (4.32) applies to $v, \tilde{\varphi}$, and we get

$$\|v, \tilde{\varphi}\|_{T,s} \leq C_s(\|\tilde{g}_2 - w(0)\|_s + \delta(s)\|\tilde{g}_2 - w(0)\|_0). \quad (4.34)$$

We estimate $\|\tilde{g}_2 - w(0)\|_s \leq \|\tilde{g}_2\|_s + \|w\|_{T,s}$, we use (4.33) and $\|\tilde{h}\|_{T,s} \leq \|v\|_{T,s} + \|w\|_{T,s}$ to conclude that

$$\|\tilde{h}, \tilde{\varphi}\|_{T,s} \leq C_s\{\|\tilde{g}\|_{T,s} + \delta(s)\|\tilde{g}\|_{T,0}\} \quad (4.35)$$

where we have denoted, in short, $\|\tilde{g}\|_{T,s} := \|\tilde{g}_1\|_{T,s} + \|\tilde{g}_2\|_s + \|\tilde{g}_3\|_s$. This proves the higher regularity for the transformed control problem (4.17). By the definitions in (4.14),

$$\begin{aligned} \|\varphi\|_{T,s} &\leq C_s(\|\tilde{\varphi}\|_{T,s} + \delta(s)\|\tilde{\varphi}\|_{T,0}), & \|h\|_{T,s} &\leq C_s(\|\tilde{h}\|_{T,s} + \delta(s)\|\tilde{h}\|_{T,0}), \\ \|\tilde{g}\|_{T,s} &\leq C_s(\|g\|_{T,s} + \delta(s)\|g\|_{T,0}), \end{aligned}$$

and the proof of (4.9) is complete.

The bound (4.10) is deduced in a classical way from the fact that h, φ solve the equations $\mathcal{L}_0^* \varphi = 0, \mathcal{L}_0 h = \chi_\omega \varphi + g_1$. \square

Remark 4.4. Another possible way to prove higher regularity for h, φ is to apply the argument of [19, 32, 3] directly to the control problem for \mathcal{L}_0 , instead of passing to the transformed problem (4.17), applying that argument, and then going back to h, φ . Such a more direct method adapted to the present case would require the construction of two operators A_s, B_s such that

- (i) $C_1\|v\|_s \leq \|A_s v\|_0 \leq C_2\|v\|_s$ (equivalent norm in H^s),
- (ii) the commutator $[\mathcal{L}_0, A_s]$ is an operator of order $s - 1$,
- (iii) the difference $B_s \mathcal{L}_0^* - \mathcal{L}_0^* A_s$ is also of order $s - 1$.

The construction of such A_s, B_s is possible, but probably the proof given above is more straightforward, and it fully exploits the advantages of conjugating \mathcal{L}_0 to \mathcal{L}_5 (Section 2). The main point is that the commutator $[\mathcal{L}_5, \Lambda^s]$ is of order $s - 1$ (because \mathcal{L}_5 has constant coefficients up to a *bounded* remainder), while $[\mathcal{L}_0, \Lambda^s]$ is of order $s + 2$ (because \mathcal{L}_0 , which was obtained by linearizing a *quasi-linear* PDE, has variable coefficients also at the highest order), so that a modified version A_s of Λ^s is needed. \square

In view of the application of Nash-Moser theorem in section 5, we define the spaces

$$E_s := X_s \times X_s, \quad X_s := C([0, T], H_x^{s+6}) \cap C^1([0, T], H_x^{s+3}) \cap C^2([0, T], H_x^s) \quad (4.36)$$

and

$$F_s := \{g = (g_1, g_2, g_3) : g_1 \in C([0, T], H_x^{s+6}) \cap C^1([0, T], H_x^s), g_2, g_3 \in H_x^{s+6}\} \quad (4.37)$$

equipped with the norms

$$\|u, f\|_{E_s} := \|u\|_{X_s} + \|f\|_{X_s}, \quad \|u\|_{X_s} := \|u\|_{T, s+6} + \|\partial_t u\|_{T, s+3} + \|\partial_{tt} u\|_{T, s} \quad (4.38)$$

and

$$\|g\|_{F_s} := \|g_1\|_{T, s+6} + \|\partial_t g_1\|_{T, s} + \|g_2, g_3\|_{s+6}. \quad (4.39)$$

With this notation, we have proved the following linear inversion result.

Theorem 4.5 (Right inverse of the linearized operator). *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. There exist two positive universal constants σ, τ and a positive constant δ_* depending on T, ω with the following property.*

Let $s \in [0, r - \tau]$, where r is the regularity of the nonlinearity \mathcal{N} (see Lemma 2.1). Let $g = (g_1, g_2, g_3) \in F_s$, and let $(u, f) \in E_{s+\sigma}$, with $\|u\|_{X_\sigma} \leq \delta_$. Then there exists $(h, \varphi) := \Psi(u, f)[g] \in E_s$ such that*

$$P'(u)[h] - \chi_\omega \varphi = g_1, \quad h(0) = g_2, \quad h(T) = g_3, \quad (4.40)$$

and

$$\|h, \varphi\|_{E_s} \leq C_s (\|g\|_{F_s} + \|u\|_{X_{s+\sigma}} \|g\|_{F_0}) \quad (4.41)$$

where C_s depends on s, T, ω .

5 Proofs

In this section we prove Theorems 1.1 and 1.4.

5.1 Proof of Theorem 1.1

The spaces defined in (4.36)-(4.39), with $s \geq 0$, form scales of Banach spaces, with compact embedding $F_b \hookrightarrow F_a$ for $b > a$.

We define smoothing operators S_θ in the following way. We fix a C^∞ function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ with $0 \leq \varphi \leq 1$,

$$\varphi(\xi) = 1 \quad \forall |\xi| \leq 1 \quad \text{and} \quad \varphi(\xi) = 0 \quad \forall |\xi| \geq 2.$$

For any real number $\theta \geq 1$, let S_θ be the Fourier multiplier with symbol $\varphi(\xi/\theta)$, namely

$$S_\theta u(x) := \sum_{k \in \mathbb{Z}} \hat{u}_k \varphi(k/\theta) e^{ikx} \quad \text{where} \quad u(x) = \sum_{k \in \mathbb{Z}} \hat{u}_k e^{ikx} \in L^2(\mathbb{T}). \quad (5.1)$$

The definition of S_θ extends to functions $u(t, x) = \sum_{k \in \mathbb{Z}} \hat{u}_k(t) e^{ikx}$ depending on time in the obvious way. Since S_θ and ∂_t commute, the smoothing operators S_θ are defined on the spaces E_s, F_s defined in (4.36)-(4.37) by setting $S_\theta(u, f) := (S_\theta u, S_\theta f)$ and similarly on

$g = (g_1, g_2, g_3)$. One easily verifies that S_θ satisfies (B.1)-(B.4) on E_s and F_s . We define the spaces E'_a with norm $\|\cdot\|'_a$ and F'_b with $\|\cdot\|'_b$ as constructed in section B.

We observe that $\Phi(u, f) := (P(u) - \chi_\omega f, u(0), u(T))$ defined in (1.12)-(1.13) belongs to F_s when $(u, f) \in E_{s+3}$, $s \in [0, r-6]$, with $\|u\|_{T,4} \leq 1$. Its second derivative is

$$\Phi''(u, f)[(h_1, \varphi_1), (h_2, \varphi_2)] = \begin{pmatrix} P''(u)[h_1, h_2] \\ 0 \\ 0 \end{pmatrix}.$$

For u in a fixed ball $\|u\|_{X_1} \leq C$, with C small enough, we estimate

$$\|P''(u)[h, w]\|_{F_s} \leq C_s (\|h\|_{X_1} \|w\|_{X_{s+3}} + \|h\|_{X_{s+3}} \|w\|_{X_1} + \|u\|_{X_{s+3}} \|h\|_{X_1} \|w\|_{X_1}) \quad (5.2)$$

for all $s \in [0, r-6]$. We fix $\delta, \beta, \alpha, a_1, a_2, V$ such that

$$\delta > 0, \quad \beta = \alpha > \max\{\sigma, 4 + \delta\}, \quad a_1 = 1, \quad (5.3)$$

$$\alpha + \delta + 3 \leq a_2 \leq r - \tau, \quad V = \{v \in E'_\alpha : \|v\|_{E_\sigma} \leq \delta_*\} \quad (5.4)$$

where the constants σ, τ, δ_* are given in Theorem 4.5. Bound (5.2) with $s = \beta + \delta$ gives (B.9) where $(m'_j, m''_j, m'''_j) = (\beta + \delta + 3, 1, 1)$ or $(1, \beta + \delta + 3, 1)$ or $(1, 1, \beta + \delta + 3)$. As a consequence, (B.11) is satisfied. Moreover (4.41) with $s = a \in [a_1, a_2]$ gives (B.10). Note that V is an E'_α convex neighborhood of zero because, since $\sigma < \alpha$, the $\|\cdot\|_\sigma$ norm is weaker than $\|\cdot\|'_\alpha$.

The right inverse Ψ in Theorem 4.5 maps the linearization point (u, f) and the datum g to the solution $(h, \varphi) = \Psi(u, f)[g]$ of (4.40). Since Ψ is independent of f and linear in g with bound (4.41), we only have to check that the map $u \mapsto (h, \varphi)$ is continuous from $u \in X_\infty$, $\|u\|_{X_\sigma} \leq \delta_*$ to $(h, \varphi) \in E_{a_2}$. The continuous dependence of the coefficients $a_i(u)$ in (2.1) on u , with some loss of derivatives in x (like in (2.3)), is easy to check directly. Now we adapt an argument from [3, Lemma 8.2-(iv)]. Let $(h, \varphi) = \Psi(u, f)[g]$ and $(h', \varphi') = \Psi(u', f)[g]$. We have proved in Lemma 4.1 that

$$\mathcal{L}_0(u)^* \varphi = 0, \quad \mathcal{L}_0(u)h = \chi_\omega \varphi + g_1, \quad h(0) = g_2, \quad h(T) = g_3$$

and

$$\mathcal{L}_0(u')^* \varphi' = 0, \quad \mathcal{L}_0(u')h' = \chi_\omega \varphi' + g_1, \quad h'(0) = g_2, \quad h'(T) = g_3.$$

Let φ'', h'' be the solutions of the Cauchy problems

$$\begin{cases} \mathcal{L}_0(u)^* \varphi'' = 0 \\ \varphi''(T) = \varphi'(T) \end{cases} \quad \begin{cases} \mathcal{L}_0(u)h'' = \chi_\omega \varphi'' + g_1 \\ h''(T) = g_3 \end{cases}$$

and define $h''_{in} := h''(0)$. By Lemma 4.1, $(h'', \varphi'') = \Psi(u, f)[(g_1, h''_{in}, g_3)]$. Since $\Psi(u, f)$ is linear in g , we get

$$(h - h'', \varphi - \varphi'') = \Psi(u, f)[(0, g_2 - h''_{in}, 0)].$$

Hence, by (4.41),

$$\|h - h'', \varphi - \varphi''\|_{E_s} \leq \|u\|_{X_{s+\sigma}} \|g_2 - h''_{in}\|_{s+6}. \quad (5.5)$$

On the other hand, the difference $\varphi'' - \varphi'$ satisfies

$$\mathcal{L}_0(u)^*(\varphi'' - \varphi') = F_0, \quad (\varphi'' - \varphi')(T) = 0 \quad \text{where} \quad F_0 := (\mathcal{L}_0(u')^* - \mathcal{L}_0(u)^*)\varphi'.$$

Hence, by Lemma 4.1, $\|\varphi'' - \varphi'\|_{T,s} \leq C\|F_0\|_{T,s} \leq C\|u - u'\|_{T,s+\sigma}\|\varphi'\|_{T,s+3}$.

Now we estimate $g_2 - h''_{in}$, which is the value of $h' - h''$ at $t = 0$. The difference $h' - h''$ satisfies

$$\mathcal{L}_0(u')(h' - h'') = F_1, \quad (h' - h'')(T) = 0 \quad \text{where} \quad F_1 := (\mathcal{L}_0(u) - \mathcal{L}_0(u'))h'' + \chi_\omega(\varphi' - \varphi'').$$

We deduce that $\|h' - h''\|_{T,s} \leq C(\|u - u'\|_{T,s+\sigma}\|h''\|_{T,s+3} + \|\varphi' - \varphi''\|_{T,s})$. Hence, by triangular inequality, $\|\varphi - \varphi'\|_{T,s} \leq \|\varphi - \varphi''\|_{T,s} + \|\varphi'' - \varphi'\|_{T,s}$ vanishes when $u' \rightarrow u$ in $X_{s+\sigma}$.

The difference $\|\partial_t^k \varphi - \partial_t^k \varphi'\|_{T,s-3k}$, $k = 1, 2$ is estimated similarly. The estimate for $\|h - h'\|_{X_s}$ is deduced from the one for $\|\varphi - \varphi'\|_{X_s}$ using Lemma A.7 and Remark A.8. This concludes the proof of the continuity of Ψ as required by Theorem B.2.

Now that all the hypotheses of Theorem B.2 have been checked, from Theorem B.2 and Remarks B.3-B.4 we obtain that, if $w = (0, u_{in}, u_{end}) \in F'_\beta$ with $\|w\|'_{F_\beta}$ small enough, then there exists a solution $(u, f) \in E'_\alpha$ of the equation $\Phi(u, f) = w$, with $\|u, f\|'_{E_\alpha} \leq C\|w\|'_{F_\beta}$ (and recall that $\beta = \alpha$). We fix $s_1 := \alpha + 6$, and (1.10) is proved. In fact, we have proved slightly more than (1.10), because $\|w\|'_{F_\beta} \leq C\|w\|_{F_\beta}$ and $\|u, f\|_{E_\mu} \leq C_\mu\|u, f\|'_{E_\alpha}$ for all $\mu < \alpha$.

We have found a solution (u, f) of the control problem (1.8)-(1.9). Now we prove that u is the unique solution of the Cauchy problem (1.8), with that given f . Let u, v be two solutions of (1.8) in E_{s-6} for all $s < s_1$. We calculate

$$P(u) - P(v) = \int_0^1 P'(v + \lambda(u - v))[u - v] d\lambda =: \bar{\mathcal{L}}_0[u - v]$$

where

$$\begin{aligned} \bar{\mathcal{L}}_0 &:= \partial_t + (1 + \bar{a}_3(t, x))\partial_{xxx} + \bar{a}_2(t, x)\partial_{xx} + \bar{a}_1(t, x)\partial_x + \bar{a}_0(t, x), \\ \bar{a}_i(t, x) &:= \int_0^1 a_i(v + \lambda(u - v))(t, x) d\lambda, \quad i = 0, 1, 2, 3. \end{aligned}$$

and $a_i(u)$ is defined in (2.2). Note that $\bar{a}_2 = 2\partial_x \bar{a}_3$ because $a_2(v + \lambda(u - v)) = 2\partial_x a_3(v + \lambda(u - v))$ for all $\lambda \in [0, 1]$. The difference $u - v$ satisfies $\bar{\mathcal{L}}_0(u - v) = 0$, $(u - v)(0) = 0$. Hence, by Lemma A.7, $u - v = 0$. The proof of Theorem 1.1 is complete.

5.2 Proof of Theorem 1.4

We define

$$E_s := C([0, T], H_x^{s+6}) \cap C^1([0, T], H_x^{s+3}) \cap C^2([0, T], H_x^s), \quad (5.6)$$

$$F_s := \{g = (g_1, g_2) : g_1 \in C([0, T], H_x^{s+6}) \cap C^1([0, T], H_x^s), g_2 \in H_x^{s+6}\} \quad (5.7)$$

equipped with norms

$$\|u\|_{E_s} := \|u\|_{T,s+6} + \|\partial_t u\|_{T,s+3} + \|\partial_{tt} u\|_{T,s} \quad (5.8)$$

$$\|g\|_{F_s} := \|g_1\|_{T,s+6} + \|\partial_t g_1\|_{T,s} + \|g_2\|_{s+6}, \quad (5.9)$$

and $\Phi(u) := (P(u), u(0))$. Given $w := (f, u_{in}) \in F_{s_0}$, the Cauchy problem (1.17) writes $\Phi(u) = w$. We fix $\delta, \beta, \alpha, a_1, a_2, V$ like in (5.3)-(5.4), where the constants σ, δ_* are given in Lemma A.7 and $\tau = 3$ by Lemma 2.1. Assumption (B.10) about the right inverse of the linearized operator is satisfied by Lemmas A.7 and 2.1. The proof that all the other assumptions of Theorem B.2 hold follows from the proof of Theorem 1.1. We fix $s_0 := \alpha + 6$. Then Theorem B.2 applies, giving the existence part of Theorem 1.4. The uniqueness of the solution is proved exactly as in the proof of Theorem 1.1. \square

A Well-posedness of linear operators

Lemma A.1. *Let $T > 0$, $m \in \mathbb{R}$, $s \in \mathbb{R}$, $f \in C([0, T], H_x^s)$, with $f(t, x) = \sum_{n \in \mathbb{Z}} f_n(t) e^{inx}$. Let A be the linear operator defined by $Af := v$ where v is the solution of*

$$\begin{cases} \partial_t v + m \partial_{xxx} v = f & \forall (t, x) \in [0, T] \times \mathbb{T}, \\ v(0, x) = 0. \end{cases} \quad (\text{A.1})$$

Then

$$Af(t, x) = \sum_{n \in \mathbb{Z}} (Af)_n(t) e^{inx}, \quad (Af)_n(t) = \int_0^t e^{imn^3(\tau-t)} f_n(\tau) d\tau, \quad (\text{A.2})$$

Af belongs to $C([0, T], H_x^s) \cap C^1([0, T], H_x^{s-3})$, and

$$\|Af\|_{T,s} \leq T \|f\|_{T,s}. \quad (\text{A.3})$$

Proof. Formula (A.2) simply comes from variation of constants. By Hölder's inequality,

$$|(Af)_n(t)| \leq \sqrt{t} \left(\int_0^t |f_n(\tau)|^2 d\tau \right)^{\frac{1}{2}} \quad \forall t \in [0, T]$$

and therefore, for each $t \in [0, T]$,

$$\begin{aligned} \|Af(t)\|_{H_x^s}^2 &= \sum_{n \in \mathbb{Z}} |(Af)_n(t)|^2 \langle n \rangle^{2s} \leq \sum_{n \in \mathbb{Z}} t \int_0^t |f_n(\tau)|^2 d\tau \langle n \rangle^{2s} \\ &\leq t \int_0^t \sum_{n \in \mathbb{Z}} |f_n(\tau)|^2 \langle n \rangle^{2s} d\tau = t \int_0^t \|f(\tau)\|_{H_x^s}^2 d\tau \leq t^2 \|f\|_{C([0,t], H_x^s)}^2. \end{aligned}$$

Taking the sup over $t \in [0, T]$ we get the thesis. \square

We remark that for $s \leq 3$ the operator A is well-defined in the sense of distributions. We also recall that $\mathcal{L}(H_x^s)$ is the space of linear bounded operators of H_x^s into itself, with operator norm $\|L\|_{\mathcal{L}(H_x^s)} := \sup\{\|Lh\|_s : h \in H_x^s, \|h\|_s = 1\}$.

Lemma A.2. (i) (LWP). *Let $T > 0$, $s \in \mathbb{R}$, $\mathcal{R} \in C([0, T], \mathcal{L}(H_x^s))$, and let*

$$r_s := \|\mathcal{R}\|_{C([0,T], \mathcal{L}(H_x^s))} = \sup_{t \in [0,T]} \|\mathcal{R}(t)\|_{\mathcal{L}(H_x^s)}, \quad \mathcal{L}_5 := \partial_t + m \partial_{xxx} + \mathcal{R}. \quad (\text{A.4})$$

Let $\alpha \in H_x^s$ and $f \in C([0, T], H_x^s)$. If $Tr_s \leq 1/2$, then the Cauchy problem

$$\begin{cases} \mathcal{L}_5 u = f \\ u(0, x) = \alpha(x) \end{cases} \quad (\text{A.5})$$

has a unique solution $u \in C([0, T], H_x^s)$. The solution u satisfies

$$\|u\|_{T,s} \leq (1 + 2Tr_s) \|\alpha\|_s + 2T \|f\|_{T,s} \leq 2(\|\alpha\|_s + T \|f\|_{T,s}). \quad (\text{A.6})$$

(ii) (Tame LWP). *Let $T > 0$, $s \in \mathbb{R}$, $s_1 \in \mathbb{R}$ with $s \geq s_1$, and let $\mathcal{R} \in C([0, T], \mathcal{L}(H_x^s)) \cap C([0, T], \mathcal{L}(H_x^{s_1}))$. Assume that*

$$\|\mathcal{R}(t)h\|_s \leq c_1 \|h\|_s + c_s \|h\|_{s_1}, \quad \|\mathcal{R}(t)h\|_{s_1} \leq c_1 \|h\|_{s_1} \quad \forall h \in H_x^s, \quad (\text{A.7})$$

for all $t \in [0, T]$, where c_1, c_s are positive constants. Let $\alpha \in H_x^s$. If

$$Tc_1 \leq 1/2, \quad (\text{A.8})$$

then the solution $u \in C([0, T], H_x^{s_1})$ of the Cauchy problem (A.5) given in (i) belongs to $C([0, T], H_x^s)$, with

$$\|u\|_{T,s} \leq 2T\|f\|_{T,s} + (1 + 2Tc_1)\|\alpha\|_s + 4Tc_s(T\|f\|_{T,s_1} + \|\alpha\|_{s_1}). \quad (\text{A.9})$$

(iii) (GWP). Let $T > 0$, $s \in \mathbb{R}$, $\mathcal{R} \in C([0, T], \mathcal{L}(H_x^s))$, and let r_s be defined in (A.4). Let $\alpha \in H_x^s$. Then the Cauchy problem (A.5) has a unique global solution $u \in C([0, T], H_x^s)$, with

$$\|u\|_{T,s} \leq 2^{4Tr_s}(\|\alpha\|_s + 4T\|f\|_{T,s}). \quad (\text{A.10})$$

(iv) (Tame GWP). Let $T > 0$, $s \in \mathbb{R}$, $s_1 \in \mathbb{R}$ with $s \geq s_1$, and let $\mathcal{R} \in C([0, T], \mathcal{L}(H_x^s)) \cap C([0, T], \mathcal{L}(H_x^{s_1}))$. Assume that (A.7) holds for all $t \in [0, T]$, where c_1, c_s are positive constants. Let $\alpha \in H_x^s$. Then the global solution $u \in C([0, T], H_x^s)$ of the Cauchy problem (A.5) given in (iii) satisfies

$$\|u\|_{T,s} \leq 2^{4Tc_1}(\|\alpha\|_s + 4Tc_s\|\alpha\|_{s_1} + 2T\|f\|_{T,s} + 4T^2c_s\|f\|_{T,s_1}). \quad (\text{A.11})$$

Proof. (i) Write $u = v + w$, where $v(t, x)$ is the solution of

$$\partial_t v + m\partial_{xxx}v = 0, \quad v(0, x) = \alpha(x). \quad (\text{A.12})$$

Hence u solves (A.5) if and only if $w(t, x)$ solves

$$\partial_t w + m\partial_{xxx}w + \mathcal{R}w = -\mathcal{R}v + f, \quad w(0, x) = 0. \quad (\text{A.13})$$

By Lemma A.1, (A.13) is the fixed point problem

$$w = \Psi(w), \quad (\text{A.14})$$

where $\Psi(w) := A[f - \mathcal{R}(v + w)]$. Let $B_\rho := \{w \in C([0, T], H_x^s) : \|w\|_{T,s} \leq \rho\}$, $\rho \geq 0$. Then

$$\|\Psi(w)\|_{T,s} \leq T(\|f\|_{T,s} + r_s\|\alpha\|_s + r_s\rho), \quad \|\Psi(w_1) - \Psi(w_2)\|_{T,s} \leq Tr_s\|w_1 - w_2\|_{T,s} \quad (\text{A.15})$$

for all $w, w_1, w_2 \in B_\rho$. By assumption, $Tr_s \leq 1/2$. Therefore, for any $\rho \geq 2T(\|f\|_{T,s} + r_s\|\alpha\|_s)$, Ψ is a contraction in B_ρ . In particular, we fix $\rho = \rho_0 := 2T(\|f\|_{T,s} + r_s\|\alpha\|_s)$. Hence there exists a fixed point $w \in B_{\rho_0}$ of Ψ , with $\|w\|_{T,s} \leq \rho_0 \leq 2T\|f\|_{T,s} + \|\alpha\|_s$. As a consequence, there exists a solution $u \in C([0, T], H_x^s)$ of (A.5) with $\|u\|_{T,s} \leq 2(T\|f\|_{T,s} + \|\alpha\|_s)$. By the contraction lemma, the solution u is unique in any ball B_ρ , $\rho \geq \rho_0$, and therefore it is unique in $C([0, T], H_x^s)$.

(ii) By assumption, $Tc_1 \leq 1/2$, and therefore, by (i), there exists a unique solution $u \in C([0, T], H_x^{s_1})$. It remains to prove that u satisfies (A.9). By construction, $u = v + w$, where $v \in C([0, T], H_x^s)$ is the solution of (A.12), with $\|v(t)\|_s = \|\alpha\|_s$ for all $t \in [0, T]$, and $w \in C([0, T], H_x^{s_1})$ solves (A.14). By the iterative scheme of the contraction lemma, w is the limit in $C([0, T], H_x^{s_1})$ of the sequence (w_n) , where $w_0 := 0$, and $w_{n+1} := \Psi(w_n)$ for all $n \in \mathbb{N}$. By (A.7) and (A.3), Ψ maps $C([0, T], H_x^s)$ into itself, therefore $w_n \in C([0, T], H_x^s)$ for all $n \geq 0$. Let $h_n := w_n - w_{n-1}$, $n \geq 1$, so that $w_n = \sum_{k=1}^n h_k$. One has $h_{n+1} = -A\mathcal{R}h_n$ for all $n \geq 1$, and

$$\|h_{n+1}\|_{T,s} \leq Tc_1\|h_n\|_{T,s} + Tc_s\|h_n\|_{T,s_1}, \quad \|h_{n+1}\|_{T,s_1} \leq Tc_1\|h_n\|_{T,s_1}, \quad \forall n \geq 1.$$

Hence, by induction, for all $n \geq 1$ we have

$$\begin{aligned} \|h_n\|_{T,s} &\leq (Tc_1)^{n-1}\|h_1\|_{T,s} + (n-1)(Tc_1)^{n-2}Tc_s\|h_1\|_{T,s_1}, \\ \|h_n\|_{T,s_1} &\leq (Tc_1)^{n-1}\|h_1\|_{T,s_1}. \end{aligned} \quad (\text{A.16})$$

Also, $\|h_1\|_{T,s} \leq T\|f\|_{T,s} + Tc_1\|\alpha\|_s + Tc_s\|\alpha\|_{s_1}$ and $\|h_1\|_{T,s_1} \leq T\|f\|_{T,s_1} + Tc_1\|\alpha\|_{s_1}$. Therefore

$$\begin{aligned} \|h_n\|_{T,s} &\leq (Tc_1)^{n-1}T\|f\|_{T,s} + (Tc_1)^n\|\alpha\|_s + (n-1)(Tc_1)^{n-2}Tc_sT\|f\|_{T,s_1} \\ &\quad + n(Tc_1)^{n-1}Tc_s\|\alpha\|_{s_1}, \\ \|h_n\|_{T,s_1} &\leq (Tc_1)^{n-1}T\|f\|_{T,s_1} + (Tc_1)^n\|\alpha\|_{s_1} \quad \forall n \geq 1. \end{aligned} \quad (\text{A.17})$$

Since $Tc_1 \leq 1/2$, the sequence $w_n = \sum_{k=1}^n h_k$ converges in $C([0, T], H_x^s)$ to some limit $\tilde{w} \in C([0, T], H_x^s)$. Since w_n converges to w in $C([0, T], H_x^{s_1})$, the two limits coincide, and $w \in C([0, T], H_x^s)$. Since $\|w\|_{T,s} \leq \sum_{k=1}^\infty \|h_k\|_{T,s}$, we get

$$\|w\|_{T,s} \leq 2T(\|f\|_{T,s} + c_1\|\alpha\|_s) + 4Tc_s(T\|f\|_{T,s_1} + \|\alpha\|_{s_1}). \quad (\text{A.18})$$

Since $u = v + w$, we deduce (A.9).

(iii). If $Tr_s \leq 1/2$, the result is given by (i). Let $Tr_s > 1/2$, and fix $N \in \mathbb{N}$ such that $2Tr_s \leq N \leq 4Tr_s$. Let $T_0 := T/N$, so that $1/4 \leq T_0r_s \leq 1/2$. Divide the interval $[0, T]$ in the union $I_1 \cup \dots \cup I_N$, where $I_n := [(n-1)T_0, nT_0]$. Applying (i) on the time interval $I_1 = [0, T_0]$ gives the solution $u_1 \in C(I_1, H_x^s)$, with $\|u_1\|_{C(I_1, H_x^s)} \leq b\|\alpha\|_s + 2T_0\|f\|_{T,s}$, where $b := 1 + 2T_0r_s$. Now consider the Cauchy problem on I_2 with initial datum $u(T_0) = u_1(T_0)$. Applying (i) on I_2 gives the solution $u_2 \in C(I_2, H_x^s)$, with

$$\|u_2\|_{C(I_2, H_x^s)} \leq b\|u_1(T_0)\|_s + 2T_0\|f\|_{T,s} \leq b^2\|\alpha\|_s + (1+b)2T_0\|f\|_{T,s}.$$

We iterate the procedure N times. At the last step, we find the solution u_N defined on I_N , with $\|u_N\|_{C(I_N, H_x^s)} \leq b^N\|\alpha\|_s + (b^N - 1)\frac{1}{b-1}2T_0\|f\|_{T,s}$. We define $u(t) := u_n(t)$ for $t \in I_n$, and the thesis follows, using that $b \leq 2$.

(iv) If $Tc_1 \leq 1/2$, the result is given by (ii). Let $Tc_1 > 1/2$, and fix $N \in \mathbb{N}$ such that $2Tc_1 \leq N \leq 4Tc_1$. Let $T_0 := T/N$, so that $1/4 \leq T_0c_1 \leq 1/2$. Split $[0, T] = I_1 \cup \dots \cup I_N$, where $I_n := [(n-1)T_0, nT_0]$. Perform the same procedure as above. Using (A.9), and $1 + 2T_0c_1 \leq 2$, by induction we get

$$\begin{aligned} \|u_n\|_{C(I_n, H_x^s)} &\leq 2^n\|\alpha\|_s + (2^n - 1)2T_0\|f\|_{T,s} + n2^{n-1}4T_0c_s\|\alpha\|_{s_1} \\ &\quad + [2^n(n-1) + 1]4T_0c_sT_0\|f\|_{T,s_1}, \\ \|u_n\|_{C(I_n, H_x^{s_1})} &\leq 2^n\|\alpha\|_{s_1} + (2^n - 1)2T_0\|f\|_{T,s_1}. \end{aligned}$$

This implies (A.11), recalling that $T_0c_1 \leq 1/2$ and also $NT_0 = T$, $N \geq 1$. \square

Lemma A.3. *There exist universal positive constants σ, δ_* with the following properties. Let $s \geq 0$, let $m \geq 1/2$, and let $a_{14}(t, x), a_{15}(t, x)$ be two functions with $a_{14}, \partial_t a_{14}, a_{15} \in C([0, T], H_x^{s+\sigma})$ and $\int_{\mathbb{T}} a_{14}(t, x) dx = 0$, and let $\mathcal{L}_4 := \partial_t + m\partial_{xxx} + a_{14}\partial_x + a_{15}$. Let*

$$\delta(\mu) := \|a_{14}, \partial_t a_{14}, a_{15}\|_{T, \mu+\sigma} \quad \forall \mu \in [0, s].$$

Assume $\delta(0) \leq \delta_*$. Let $f \in C([0, T], H_x^s)$, $\alpha \in H_x^s$. Then the Cauchy problem

$$\begin{cases} \mathcal{L}_4 u = f \\ u(0) = \alpha \end{cases} \quad (\text{A.19})$$

admits a unique solution $u \in C([0, T], H_x^s)$, with

$$\|u\|_{T,s} \leq C_s \{ \|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0) \} . \quad (\text{A.20})$$

Proof. Following the procedure given in Section 2.5, we define $\mathcal{S} := I + \gamma(t, x)\partial_x^{-1}$ (see (2.53)) with $\gamma(t, x) := -\frac{1}{3m}\partial_x^{-1}a_{14}(t, x)$. We have that u solves (A.19) if and only if $\tilde{u} := \mathcal{S}^{-1}u$ satisfies

$$\begin{cases} \mathcal{L}_5 \tilde{u} = \tilde{f} \\ \tilde{u}(0) = \tilde{\alpha} \end{cases}$$

where $\tilde{f} := \mathcal{S}^{-1}f$, $\tilde{\alpha} := \mathcal{S}^{-1}(0)\alpha$ and $\mathcal{L}_5 = \partial_t + m\partial_{xxx} + \mathcal{R}$, with $\mathcal{R} = \mathcal{S}^{-1}\{a_{15} + (a_{14}\gamma - (a_{14})_x)\pi_0 + (\mathcal{L}_4\gamma)\partial_x^{-1}\}$. Then the thesis follows as a straightforward consequence of Lemma A.2 and Lemma 2.7. \square

Lemma A.4. *There exist universal positive constants σ, δ_* with the following properties. Let $s \geq 0$, let $m \geq 1/2$, and let $a_{12}(t, x), a_{13}(t, x)$ be two functions with $a_{12}, \partial_t a_{12}, a_{13} \in C([0, T], H_x^{s+\sigma})$, and let $\mathcal{L}_3 := \partial_t + m\partial_{xxx} + a_{12}\partial_x + a_{13}$. Let*

$$\delta(\mu) := \|a_{12}, \partial_t a_{12}, a_{13}\|_{T, \mu+\sigma} \quad \forall \mu \in [0, s].$$

Assume $\delta(0) \leq \delta_*$. Let $f \in C([0, T], H_x^s)$, $\alpha \in H_x^s$. Then the Cauchy problem

$$\begin{cases} \mathcal{L}_3 u = f \\ u(0) = \alpha \end{cases} \quad (\text{A.21})$$

admits a unique solution $u \in C([0, T], H_x^s)$, with

$$\|u\|_{T,s} \leq C_s \{ \|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0) \} . \quad (\text{A.22})$$

Proof. Following the procedure given in Section 2.4, we define $\mathcal{T}h(t, x) := h(t, x + p(t))$ (see (2.46)) with $p(t) := -\frac{1}{2\pi} \int_0^t \int_{\mathbb{T}} a_{12}(s, x) dx ds$. We have that u solves (A.21) if and only if $\tilde{u} := \mathcal{T}^{-1}u$ satisfies

$$\begin{cases} \mathcal{L}_4 \tilde{u} = \tilde{f} \\ \tilde{u}(0) = \alpha \end{cases}$$

(note that $\mathcal{T}(0)$ is the identity) where $\tilde{f} := \mathcal{T}^{-1}f$, and $\mathcal{L}_4 = \partial_t + m\partial_{xxx} + a_{14}\partial_x + a_{15}$, with a_{14}, a_{15} given by formula (2.48). Then the thesis follows as a straightforward consequence of Lemma A.3 and Lemma 2.6. \square

Lemma A.5. *There exist universal positive constants σ, δ_* with the following properties. Let $s \geq 0$, let $m \geq 1/2$, and let $a_8(t, x), a_9(t, x), a_{10}(t, x)$ be three functions with $a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10} \in C([0, T], H_x^{s+\sigma})$ and $\int_{\mathbb{T}} a_8(t, x) dx = 0$, and let $\mathcal{L}_2 := \partial_t + m\partial_{xxx} + a_8\partial_{xx} + a_9\partial_x + a_{10}$. Let*

$$\delta(\mu) := \|a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10}\|_{T, \mu+\sigma} \quad \forall \mu \in [0, s].$$

Assume $\delta(0) \leq \delta_*$. Let $f \in C([0, T], H_x^s)$, $\alpha \in H_x^s$. Then the Cauchy problem

$$\begin{cases} \mathcal{L}_2 u = f \\ u(0) = \alpha \end{cases} \quad (\text{A.23})$$

admits a unique solution $u \in C([0, T], H_x^s)$, with

$$\|u\|_{T,s} \leq C_s \{ \|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0) \}. \quad (\text{A.24})$$

Proof. Following the procedure given in Section 2.3, we define $\mathcal{M}h(t, x) := q(t, x)h(t, x)$ (see (2.37)) with $q(t, x) := \exp\{-\frac{1}{3m}(\partial_x^{-1}a_8)(t, x)\}$. We have that u solves (A.23) if and only if $\tilde{u} := \mathcal{M}^{-1}u$ satisfies

$$\begin{cases} \mathcal{L}_3 \tilde{u} = \tilde{f} \\ \tilde{u}(0) = \tilde{\alpha} \end{cases}$$

where $\tilde{f} := \mathcal{M}^{-1}f$, $\tilde{\alpha} := \mathcal{M}^{-1}(0)\alpha$, and $\mathcal{L}_3 = \partial_t + m\partial_{xxx} + a_{12}\partial_x + a_{13}$, with a_{12}, a_{13} given by formula (2.39). Then the thesis follows as a straightforward consequence of Lemma A.4 and Lemma 2.5. \square

Lemma A.6. *There exist universal positive constants σ, δ_* with the following properties. Let $s \geq 0$ and let $a_4(t), a_5(t, x), a_6(t, x), a_7(t, x)$ be four functions with $a_4 \in C^1([0, T], \mathbb{R})$, $a_5, \partial_t a_5, a_6, \partial_t a_6, a_7 \in C([0, T], H_x^{s+\sigma})$ and $\int_{\mathbb{T}} a_5(t, x) dx = 0$, and let $\mathcal{L}_1 := \partial_t + a_4 \partial_{xxx} + a_5 \partial_{xx} + a_6 \partial_x + a_7$. Let*

$$\delta(\mu) := \sup_{t \in [0, T]} |a_4(t) - 1| + \sup_{t \in (0, T)} |a_4'(t)| + \|a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T, \mu+\sigma} \quad \forall \mu \in [0, s]. \quad (\text{A.25})$$

Assume $\delta(0) \leq \delta_*$. Let $f \in C([0, T], H_x^s)$, $\alpha \in H_x^s$. Then the Cauchy problem

$$\begin{cases} \mathcal{L}_1 u = f \\ u(0) = \alpha \end{cases} \quad (\text{A.26})$$

admits a unique solution $u \in C([0, T], H_x^s)$, with

$$\|u\|_{T,s} \leq C_s \{ \|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0) \}. \quad (\text{A.27})$$

Proof. Following the procedure given in Section 2.2, we define $\mathcal{B}h(t, x) := h(\psi(t), x)$ (see (2.25)) with $\psi(t) := \frac{1}{m} \int_0^t a_4(s) ds$, where $m := \frac{1}{T} \int_0^T a_4(t) dt$. We have that u solves (A.26) if and only if $\tilde{u} := \mathcal{B}^{-1}u$ satisfies

$$\begin{cases} \mathcal{L}_2 \tilde{u} = \tilde{f} \\ \tilde{u}(0) = \alpha \end{cases}$$

(note that $\mathcal{B}(0)$ is the identity) where $\tilde{f} := \mathcal{B}^{-1}f$, and $\mathcal{L}_2 = \partial_t + m\partial_{xxx} + a_8\partial_{xx} + a_9\partial_x + a_{10}$, with a_8, a_9, a_{10} given by formula (2.32) (see also (2.26)). Then the thesis follows as a straightforward consequence of Lemma A.5 and Lemma 2.4. \square

Lemma A.7. *There exist universal positive constants σ, δ_* with the following properties. Let $s \geq 0$ and let $a_3(t, x), a_2(t, x), a_1(t, x), a_0(t, x)$ be four functions with $a_3, \partial_t a_3, \partial_{tt} a_3, a_1, \partial_t a_1, a_0 \in C([0, T], H_x^{s+\sigma})$ and $a_2 = c\partial_x a_3$ for some $c \in \mathbb{R}$. Let*

$$\delta(\mu) := \|a_3, \partial_t a_3, \partial_{tt} a_3, a_1, \partial_t a_1, a_0\|_{T, \mu+\sigma} \quad \forall \mu \in [0, s]. \quad (\text{A.28})$$

Assume $\delta(0) \leq \delta_*$. Let $\mathcal{L}_0 := \partial_t + (1 + a_3)\partial_{xxx} + a_2\partial_{xx} + a_1\partial_x + a_0$. Let $f \in C([0, T], H_x^s)$, $\alpha \in H_x^s$. Then the Cauchy problem

$$\begin{cases} \mathcal{L}_0 u = f \\ u(0) = \alpha \end{cases} \quad (\text{A.29})$$

admits a unique solution $u \in C([0, T], H_x^s)$, with

$$\|u\|_{T,s} \leq C_s \{ \|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0) \}. \quad (\text{A.30})$$

Proof. Following the procedure given in Section 2.1, we define $(\mathcal{A}h)(t, x) := h(t, x + \beta(t, x))$ (see (2.9)) with $\beta(t, x) := (\partial_x^{-1}\rho_0)(t, x)$, where ρ_0 is defined in (2.16)-(2.17). We have that u solves (A.29) if and only if $\tilde{u} := \mathcal{A}^{-1}u$ satisfies

$$\begin{cases} \mathcal{L}_1 \tilde{u} = \tilde{f} \\ \tilde{u}(0) = \tilde{\alpha} \end{cases}$$

where $\tilde{f} := \mathcal{A}^{-1}f$, $\tilde{\alpha} := \mathcal{A}^{-1}(0)\alpha$, and $\mathcal{L}_1 = \partial_t + a_4\partial_{xxx} + a_5\partial_{xx} + a_6\partial_x + a_7$, with a_4 not depending on the space variable x and with a_4, a_5, a_6, a_7 given by formula (2.14). Then the thesis follows as a straightforward consequence of Lemma A.6 and Lemma 2.3. \square

Remark A.8. Consider the operators $\mathcal{L}_0, \dots, \mathcal{L}_5$ defined in Lemmas A.2-A.7. Define

$$\begin{aligned} \mathcal{L}_0^* h &:= -\partial_t h - \partial_{xxx}[(1 + a_3)h] + \partial_{xx}(a_2 h) - \partial_x(a_1 h) + a_0 h \\ \mathcal{L}_1^* h &:= -\partial_t h - a_4 \partial_{xxx} h + \partial_{xx}(a_5 h) - \partial_x(a_6 h) + a_7 h \\ \mathcal{L}_2^* h &:= -\partial_t h - m \partial_{xxx} h + \partial_{xx}(a_8 h) - \partial_x(a_9 h) + a_{10} h \\ \mathcal{L}_3^* h &:= -\partial_t h - m \partial_{xxx} h - \partial_x(a_{12} h) + a_{13} h \\ \mathcal{L}_4^* h &:= -\partial_t h - m \partial_{xxx} h - \partial_x(a_{14} h) + a_{15} h \\ \mathcal{L}_5^* h &:= -\partial_t h - m \partial_{xxx} h + \mathcal{R}^T h. \end{aligned}$$

It is straightforward to check that Lemmas A.2-A.7 also hold when the operator \mathcal{L}_k ($k = 0, \dots, 5$) is replaced by \mathcal{L}_k^* . The crucial observation is that for all $k = 0, \dots, 5$ (see Remark 2.2 for the case $k = 0$) the operator $-\mathcal{L}_k^*$ has the same structure as \mathcal{L}_k (one might need to worsen the constants σ since the coefficients of $-\mathcal{L}_k^*$ involve space derivatives of the coefficients of \mathcal{L}_k). It is also immediate to verify that the same estimates also hold for the backward Cauchy problems

$$\begin{cases} \mathcal{L}_k u = f \\ u(T) = \alpha \end{cases} \quad \begin{cases} \mathcal{L}_k^* u = f \\ u(T) = \alpha \end{cases} \quad k = 0, \dots, 5. \quad (\text{A.31})$$

\square

B Nash-Moser theorem by Hörmander

We recall here a sharp Nash-Moser implicit function theorem, as proved by Hörmander in [26].

Let E_a , $a \geq 0$, be a decreasing family of Banach spaces with injections $E_b \hookrightarrow E_a$ of norm ≤ 1 when $b \geq a$. Set $E_\infty = \cap_{a \geq 0} E_a$ with the weakest topology making the injections

$E_\infty \hookrightarrow E_a$ continuous, and assume that we have given linear operators $S_\theta : E_0 \rightarrow E_\infty$ for $\theta \geq 1$ such that with constants C bounded, when a and b are bounded,

$$\|S_\theta u\|_b \leq C\|u\|_a \quad \text{if } b \leq a; \quad (\text{B.1})$$

$$\|S_\theta u\|_b \leq C\theta^{b-a}\|u\|_a \quad \text{if } a < b; \quad (\text{B.2})$$

$$\|u - S_\theta u\|_b \leq C\theta^{b-a}\|u\|_a \quad \text{if } a > b; \quad (\text{B.3})$$

$$\left\| \frac{d}{d\theta} S_\theta u \right\|_b \leq C\theta^{b-a-1}\|u\|_a. \quad (\text{B.4})$$

From (B.2)-(B.3) we can obtain the logarithmic convexity of the norms

$$\|u\|_{\lambda a + (1-\lambda)b} \leq C\|u\|_a^\lambda \|u\|_b^{1-\lambda} \quad \text{if } 0 < \lambda < 1. \quad (\text{B.5})$$

Let us consider the sequence $\{\theta_j\}_{j \in \mathbb{N}}$, with $1 = \theta_0 < \theta_1 < \dots \rightarrow \infty$, such that $\frac{\theta_{j+1}}{\theta_j}$ is bounded. Set $\Delta_j := \theta_{j+1} - \theta_j$ and

$$R_0 u := \frac{S_{\theta_1} u}{\Delta_0}, \quad R_j u := \frac{S_{\theta_{j+1}} u - S_{\theta_j} u}{\Delta_j}, \quad j \geq 1.$$

By (B.3) we deduce that, if $u \in E_b$ for some $b > a$, then

$$u = \sum_{j=0}^{\infty} \Delta_j R_j u \quad (\text{B.6})$$

with convergence in E_a . Moreover, (B.4) implies that, for all b ,

$$\|R_j u\|_b \leq C_{a,b} \theta_j^{b-a-1} \|u\|_a. \quad (\text{B.7})$$

Conversely, assume that $a_1 < a < a_2$, that $u_j \in E_{a_2}$ and that

$$\|u_j\|_b \leq M \theta_j^{b-a-1} \quad \text{if } b = a_1 \text{ or } b = a_2. \quad (\text{B.8})$$

By (B.5) this remains true with a constant factor on the right-hand side if $a_1 < b < a_2$, so that $u = \sum \Delta_j u_j$ converges in E_b if $b < a$.

Let E'_a be the set of all sums $u = \sum \Delta_j u_j$ with u_j satisfying (B.8) and introduce the norm $\|u\|'_a$ as the infimum of M over all such decompositions. It follows that $\|\cdot\|'_a$ is stronger than $\|\cdot\|_b$ if $a > b$, while (B.6) and (B.7) show that $\|\cdot\|'_a$ is weaker than $\|\cdot\|_a$. Moreover the space E'_a and, up to equivalence, its norm are independent of the choice of a_1 and a_2 ; E'_a is defined by (B.7) for any values of b to the left and to the right of a ; E'_a does not depend on the smoothing operators; in (B.3) we can replace $\|u\|_a$ by $\|u\|'_a$ if we take another constant, which may tend to ∞ as b approaches a (all these statements are proved in [26]).

Definition B.1. Fix $a > 0$. If (u_k) is a bounded sequence in E_a , and $u_k \rightarrow u$ in E_0 , then it follows from (B.5) that (u_k) is a Cauchy sequence in E_b for every $b < a$, so that the limit $u \in E_b$. In this case we say that $(u_k) \subseteq E_a$ is weakly convergent, and that u is the weak E_a limit of (u_k) .

By the definition of E'_a one can show that every element in E'_a is the weak E_a limit of a sequence in E_∞ .

Now let us suppose that we have another family F_a of decreasing Banach spaces with smoothing operators having the same properties as above. We use the same notation also for the smoothing operators. In addition we assume that the embedding $F_b \hookrightarrow F_a$ is compact when $b > a$.

Theorem B.2 ([26]). *Let α and β be fixed positive numbers, $[a_1, a_2]$ an interval with $0 \leq a_1 < \alpha < a_2$, V a convex E'_α neighborhood of 0 and Φ a map from $V \cap E_{a_2}$ to F_β which is twice differentiable and satisfies, for some $\delta > 0$,*

$$\|\Phi''(u; v, w)\|_{\beta+\delta} \leq C \sum_{j \in J} (1 + \|u\|_{m'_j}) \|v\|_{m''_j} \|w\|_{m'''_j} \quad (\text{B.9})$$

where J is a finite set. Also assume that $\Phi'(v)$, for $v \in E_\infty \cap V$, has a right inverse $\Psi(v)$ mapping F_∞ to E_{a_2} , that $(v, g) \rightarrow \Psi(v)g$ is continuous from $(E_\infty \cap V) \times F_\infty$ to E_{a_2} , and that

$$\|\Psi(v)g\|_a \leq C(\|g\|_{\beta+a-\alpha} + \|g\|_0 \|v\|_{\beta+a}), \quad a_1 \leq a \leq a_2. \quad (\text{B.10})$$

Let $a_2 \geq \max\{m'_j, m''_j, m'''_j : j \in J\}$, let $\alpha - \beta < a_1$ and

$$\max\{m'_j - \alpha, 0\} + \max\{m''_j, a_1\} + m'''_j < 2\alpha \quad \forall j \in J. \quad (\text{B.11})$$

For every $w \in F'_\beta$ with sufficiently small norm there exists a sequence $(u_n) \subset V \cap E_{a_2}$ which has a weak limit u in E'_α such that $\Phi(u_n)$ converges weakly in F'_β to $\Phi(0) + w$.

Remark B.3. In the case when the nonlinear operator Φ admits a continuous extension to the space E'_α , the limit of the sequence gives a solution of the equation $\Phi(u) = \Phi(0) + w$. \square

Remark B.4. The proof in [26] also shows that the solution u of the equation $\Phi(u) = \Phi(0) + w$ satisfies

$$\|u\|'_\alpha \leq C\|w\|'_\beta \quad \forall a \in [a_1, \alpha]. \quad (\text{B.12})$$

To obtain (B.12), first, like in [26], we take $g \in F'_\beta$ such that $g + T(g) = w$, where T is an operator (constructed in [26]) satisfying $\|T(g)\|'_\beta \leq C\|g\|'^2_\beta$, so that $\|g\|'_\beta \leq 2\|w\|'_\beta$ in a ball around the origin. Then bound (9) of [26] implies that $\|u\|'_\alpha \leq C_1\|g\|'_\beta$, and (B.12) is proved. \square

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